Recall: Mackey's criterion. $K \leq G$: finite group, $G \geq H$

An easy consequence on the decomposition of (W,ρ): repr. of H

$$\mathrm{Res}^G_K\left(\mathrm{Ind}^G_H(W,\rho)\right) \leftarrow \text{a repr. of } K$$

$$= \bigoplus_{\{s_i\} = K\backslash G/H} \mathrm{Ind}^K_{K_{s_i}}(\rho_{s_i}, W)$$

$\{s_i\} = K\backslash G/H$ ← a set of repr. of $K\backslash G/H$

$s_i^{-1} k s_i \in H \Updownarrow$

$K_{s_i} = \{k \in K \mid k \cdot s_i H = s_i H\} = K \cap s_i H s_i^{-1}$

$\rho_{s_i}(y) = \rho(\underbrace{s_i^{-1} y s_i}_{\in H}) \in GL(W)$, $y \in K_{s_i}$

$\Rightarrow$. $\mathrm{Ind}^G_H(W,\rho)$ is irreducible as a G-rep

$\Longleftrightarrow$ (W,ρ) is an irred repr. of H

<u>and</u> $\forall s \notin H$, the two repr. of $H_s = H \cap sHs^{-1}$ on W: $\rho|_W$ and ρ_s are <u>disjoint</u>

$y \in H_s$ acts as $\rho(y)$ and $\rho(s^{-1} y s)$

$\Updownarrow$

$$\left(\mathrm{ch}(\rho|_W), \mathrm{ch}(\rho_s)\right)_{H_s} = 0.$$

the first

Application

$N \trianglelefteq G$, $N \neq \{1\}$, (V,ψ): irreducible repr.

<u>Question</u> Is ψ induced from a repr. of a proper subgroup $H \subsetneq G$?

Consider $(V,\psi)|_N = \bigoplus_{i=1}^{r} (W_i, \chi_i)$ (W_i, χ_i) irreducible

$$= \bigoplus_{j=1}^{t} \overbrace{m_j (W_j, \psi_j)}^{\widetilde{W}_j}$$

$\bigoplus_{j=1}^{t} \widetilde{W}_j$ ← intrinsic decomp.

↑ irred, and $m_j \geq 1 \ \forall j$

$\forall s \in G$, $x \mapsto \overset{\psi_s}{\psi(s^{-1} x s)} \in GL(V)$ $\overset{\forall x \in G}{\cong} \psi(s)^{-1} \cdot \psi(x) \cdot \psi(s)$ $\Longrightarrow$ $\psi_s \cong \psi$ as G-repr.

$$\Rightarrow \bigoplus_{j=1}^{t} m_j \left(W_j, \underbrace{\psi_{j,s}}\right)$$

conjugation/twisting permutes the $\underline{\psi_j \text{ s}}$!

$$\uparrow$$

$$\underset{N}{\overset{\nearrow}{y}} \mapsto \psi_j(s^{-1} \underset{\underset{N}{\cap}}{y} s)$$

$\forall s \in G$ s send each $\widetilde{W_j}$ to a $\widetilde{W_{j'}}$.

conjugation / twisting permute the $\underline{(\widetilde{W_j}, \psi_j)}$'s

__Because__ ρ is __irred__ $\Rightarrow$ the action of G permutation on the $(\widetilde{W_j}, \psi_j)$'s are __transitive__ !

__Pick__ one of them, __say__ $(\widetilde{W_{j_0}}, \psi_{j_0})$.

__Suppose__ __$t \geq 2$__

__Then__ $H := \{ x \in G \mid (\widetilde{W_{j_0}}, \psi_{j_0})_x = (\widetilde{W_{j_0}}, \psi_{j_0}) \}$

$$\lneqq G$$

. __Have__ action of H on $\widetilde{W_{j_0}}$.

__and__ $V = \bigoplus_{y \in G/H} \widetilde{W_{j_0}}$

__i.e__ $V = \mathrm{Ind}_H^G (\widetilde{W_{j_0}})$.

Summary: Let $N \trianglelefteq G$, (V, ρ) is an irred rep. of G

Then either 1) $(V, \rho)|_N$ is isotypic

i.e. is a multiple of an irred repr. of N

or 2) $\exists\, N \leq H \lneqq G$ and an irred. repr $(\varphi, \hat{W})$ of H s.t. $(V, \rho) \cong \mathrm{Ind}_H^G (\hat{W}, \varphi)$.

Trivial example $\mathbb{Z}/3\mathbb{Z} \cong N \trianglelefteq S_3$ has an irred 2-dim^l repr. $(\mathbb{C}^2, \rho)$ "standard repr"

$(\mathbb{C}^2, \rho)|_N$ N has 3 irreducible characters, $\mathbb{1}, \chi, \bar{\chi}$

$\mathbb{Z}/3\mathbb{Z} = \{3 \text{ rotation}\}$

$$\begin{cases} \chi : 1+3\mathbb{Z} \mapsto e^{2\pi\sqrt{-1}/3} = \zeta \\ \bar{\chi} : 1+3\mathbb{Z} \mapsto e^{-2\pi\sqrt{-1}/3} = \bar{\zeta} = \zeta^{-1} \end{cases}$$

$$(\mathbb{C}^2, \rho)|_N \cong \overset{0}{n_0}\, \mathbb{1} \oplus \overset{1}{n_1}\, \chi \oplus \overset{1}{n_2}\, \bar{\chi}$$

$n_0, n_1, n_2 \in \mathbb{N}$

$n_0 + n_1 + n_2 = 2$

$(2, 0, 0) \leftarrow$ out

$\zeta + \zeta^{-1} = -1$

$\therefore$ They are roots of $x^2 + x + 1$.

$$\mathrm{ch}(\rho)|_N \begin{pmatrix} 1+3\mathbb{Z} \\ 2+3\mathbb{Z} \end{pmatrix} = -1$$

	$\overset{1}{\{1\}}$	$\overset{3}{(12)}$	$\overset{2}{(123)}$
$\mathbb{1}$	1	1	1
sgn	1	-1	1
ρ	2	0	-1

$$\mathrm{ch}(\rho)|_N = \chi + \bar{\chi}$$

$$(\chi + \bar{\chi}) \begin{pmatrix} 0+3\mathbb{Z} \\ 1+3\mathbb{Z} \\ 2+3\mathbb{Z} \end{pmatrix} \begin{matrix} = 2 \\ = -1 \\ = -1 \end{matrix}$$

$$\rho|_{N \cong \mathbb{Z}/3\mathbb{Z}} \cong \chi \oplus \chi^{-1} \qquad \chi_s = \begin{cases} \chi & \text{if } s \in N \\ \chi^{-1} & \text{if } s \notin N \end{cases}$$

$$\{ s \in S_3 \mid \chi_s = \chi \} = N$$

$$\rho \cong \mathrm{Ind}_N^{S_3}(\chi)$$

χ: either one of the two non-trivial 1-dim'l character of N.

Similarly, can show: "most of irred. char." of $D_n = (\mathbb{Z}/n\mathbb{Z}) \rtimes (\pm 1)$ are 2-dim'l

D_n: dihedral group with $2n$ elements

(± 1): $\mathrm{Ad}(-1)$ on $\mathbb{Z}/n\mathbb{Z}$ as $[-1]$

Prop: Let G be a supersolvable finite group and let (V, ρ) be an irred. repr. of G over $\mathbb{C}$. Then there exists a 1-dim'l repr. ρ of a subgroup H of G s.t. $(V, \rho) \cong \mathrm{Ind}_H^G(\rho)$

Exercise! Hint: Induction, say on the order of the supersolvable group and the "summary" above

Recall: solvable groups.

G: a finite group

i) G is solvable: $\exists$ $(1) = G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n = G$

s.t. $G_i \trianglelefteq G_{i+1}$ $\forall i$

and G_{i+1}/G_i is abelian

ii) G is <u>supersolvable</u> if $\exists$ a sequence of subgroups as above s.t. $G_i \trianglelefteq G$ $\forall i$.

iii) G is <u>nilpotent</u> if $\exists$ a sequence of subgps as above st. $G_i \trianglelefteq G$ $\forall i$

<u>AND</u> $G_{i+1}/G_i \subseteq Z(G/G_i)$ $\forall i$.

e.g. Every finite p-group is a nilpotent group.

S_3 is supersolvable but <u>not</u> nil<u>potent</u>

<u>Exer</u> Find an example of a solvable but not supersolvable finite <u>group</u>!

<u>Recall</u>:

We <u>showed</u>: $\forall$ irred character χ of G $\exists$ cyclic ~~subgroups~~ subgroups H_i, 1-dim^l characters ψ_i of H_i <u>and</u> positive rational number a_i $i=1,\dots,m$ s.t.

$$\chi = \sum_{i=1}^{m} a_i \cdot \mathrm{Ind}_{H_i}^{G}(\psi_i)$$

In particular:

$$I : \bigoplus_{\substack{\text{cyclic} \\ H \lneq G \\ \psi : H \to \mathbb{C}^\times \\ \text{1-dim}^{\text{l}} \text{ char. of } H}} \mathbb{Q} \otimes_{\mathbb{Z}} R(H) \xrightarrow{\bigoplus \operatorname{Ind}_G^H} \mathbb{Q} \otimes_{\mathbb{Z}} R(G)$$

is surjective

Another proof: Consequence of Frobenius reciprocity:

Both sides are finite dim$^{\text{l}}$ vector spaces with non-deg. pairings.

Frob. Reciprocity: The transpose I^* of the map I is given by restricting any given character to subgroups

I is surj $\Longleftrightarrow$ I^* is injective

Obvious!

$\exists$ a central function f on G. if $f|_{\text{every cyclic subgroup of } G} = 0$

then $f = 0$.