Sheaves and presheaves.

<u>Examples</u> Let X be a complex manifold (locally $\cong$ to an open subset of $\mathbb{C}^n$)
+ transition maps are biholomorphic

$n=1$: Riemann surfaces

$\rightsquigarrow$ $\forall$ open subset $U \subseteq X$

<u>Have</u> $\{$holomorphic functions on $U\}$
$= \mathcal{O}_{X,hol}(U)$

$U \rightsquigarrow \mathcal{O}_{X,hol}(U)$ is a sheaf of <u>rings</u>

(because given an open cover $(V_i)_{i\in I}$ of U
holomorphic function on U
$\longleftrightarrow$ holomorphic functions on V_i
<u>compatible</u> on $V_i \cap V_j$)

(being holomorphic is a local property)

Can define (Exercise)

sheaves of $\mathcal{O}_{X,hol}$-modules

<u>i.e.</u> $X \overset{open}{\supseteq} U \rightsquigarrow \mathcal{F}(U) \leftarrow$ an $\mathcal{O}_{X,hol}(U)$-module

(+ other properties such as finiteness, flatness)

<u>e.g.</u> $\mathcal{E}$ holomorphic vector bundle

$$\mathcal{E} \longrightarrow X$$

$X \underset{open}{\supseteq} U$ (locally isomorphic to $U \times \mathbb{C}^d$)
+ biholomorphic transition maps, linear in the second factor

$X \overset{open}{\supseteq} U \rightsquigarrow$ (holomorphic sections of $\mathcal{E}|_U = \mathcal{E} \times_X U$)

$\mathcal{F}(\mathcal{E}) \leftarrow$ a sheaf of $\mathcal{O}_{X,hol}$-modules

Cohomology / derived functors : allow you to pass from local properties to global properties.

<u>Want</u>: $\forall$ $\mathcal{E}/X$ holomorphic vector bundle
$X \Leftarrow$ cpx mfd

<u>Want</u> $H^i(X; \mathcal{E})$ $i \geq 0$.

such that

1) $H^0(X, \mathcal{E}) = \Gamma(X, \mathcal{E})$ = global sections of $\mathcal{E}$
$\Gamma(X,\mathcal{E}) = \mathcal{F}(\mathcal{E})(X)$

ii) $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$
short exact sequence of holomorphic vector bundles on X,

$$0 \to H^0(X, \mathcal{E}') \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{E}'')$$
$$\to H^1(X, \mathcal{E}') \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{E}'')$$
$$\vdots$$
$$\to H^i(X, \mathcal{E}') \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{E}'')$$
$$\to \cdots$$

long exact sequence.

How to define such a cohomology theory?

<u>One Ans.</u> use derived functors of

$$\Gamma(X, -) : \quad \mathcal{F} \longmapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$
$$(\text{sheaves of abelian groups on } X) \longrightarrow (\text{abel. groups})$$

<u>Note</u> $\Gamma(X,-)$ is left exact

$\to$ The category of sheaves of abelian groups on X has enough injectives.

Another abelian category: all left $\mathcal{O}_{X,\mathrm{hol}}$-modules

<u>i.e</u> sheaves of left $\mathcal{O}_{X,\mathrm{hol}}$-modules.

(property S) $\forall\, U \subseteq X$ open

$\forall$ open cover $(V_i)_{i\in I}$ of U

$$
\begin{tikzcd}
0 \arrow[r] & P(U) \arrow[r, "\alpha"] & \prod\limits_{i\in I} P(V_i) \arrow[r, shift left, "pr_1^*"] \arrow[r, shift right, "pr_2^*"'] & \prod\limits_{(i,j)\in I^2} P(V_i\cap V_j)
\end{tikzcd}
$$

pre-sheaf

$$t \qquad\qquad s=(s_i)_{i\in I} \rightrightarrows pr_1^*(s),\ pr_2^*(s)$$

Clearly if $\exists\, t$ s.t. $s=\alpha(t)$, then $pr_1^*(s)=pr_2^*(s)$

? $\quad pr_1^*(s)=pr_2^*(s)$

$\Longrightarrow \exists !\ t\in P(U)$ s.t. $\alpha(t)=s$

P is a sheaf if property S holds. (def^n)

In practice, often you start with a presheaf

$\rightsquigarrow$ a "sheafification functor" which turns a presheaf to a sheaf.

- Godement, "Théorie des faisceaux".
  (textbook)
- Serre, "FAC" Faisceaux Algébrique Cohérents.
  (famous paper by Serre)

(*) - Grothendieck. "Tohoku" — paper in Tohoku J. of Math, middle 1950's.
- Swan sheaves Chicago. Notes.

Basic commutative algebra
rings + modules over them
comm. rings

We know (understand) MUCH better than non-commutative ones!

"Review" localization:

$0 \notin S \subseteq R$: comm. ring

multiplicative i.e. $1 \in S$
and $s_1, s_2 \in S \Rightarrow s_1 \cdot s_2 \in S$

Can/Will construct:
1) a ring $S^{-1}R$ and a ring homom $R \to S^{-1}R$
   + suitable universal properties

2) $\forall$ R-module M, an $S^{-1}R$-module $S^{-1}M$

+ suitable universal properties

$$\underline{S^{-1}R} = S \times R / \sim$$

$s_1^{-1} \cdot x_1$ $\dashleftarrow$ $(s_1, x_1) \sim (s_2, x_2)$

iff $\exists t$ $t \cdot (s_1 x_2 - s_2 x_1) = 0$

– $S^{-1}R$ has a natural structure as a ring . . <u>and</u> $\underset{R}{\overset{\in}{x}} \longmapsto [(1, x)]$

is a ring homom.