Schur's lemma — orthogonality relations.

V, W: (finite dim^l) irred repr. of a finite group G over an algebraically closed field k s.t. $\#(G)\cdot 1_k \in k^\times$

$$\int_G : \mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_{k[G]}(V,W)$$

$$(*) \qquad T \longmapsto \frac{1}{\dim(V)}\mathrm{Tr}(T)\,\mathrm{Id}_V = \begin{cases} 0 & \text{if } V \not\cong_{k[G]} W \\ k\cdot \mathrm{Id}_V & V = W \end{cases}$$

When expressed in coordinates $\rho_{ij} \longleftrightarrow V$, $r_{\mu\nu} \longleftrightarrow W$ matrix repr. get:

(a) $V \not\cong_{k[G]} W \Longrightarrow \displaystyle\sum_{t\in G} r_{\mu\nu}(t)\cdot \rho_{ij}(t^{-1}) = 0$

$\forall \underset{1,\dots,\dim(W)}{\mu,\nu}, \forall \underset{1,\dots,\dim(V)}{i,j}$

(b) $V = W$

$$(*) \Longleftrightarrow \frac{1}{\#(G)} \sum_{t\in G} \sum_{\mu, i} \rho_{\nu\mu}(t)\, z_{\mu i}\, \rho_{ij}(t^{-1})$$

$$= \frac{1}{\dim(V)} \cdot \underbrace{\sum_{i=1}^{\dim(V)} z_{ii}}_{= \sum_\mu z_{\mu i}\,\delta_{i\mu}} \cdot \delta_{\nu j}$$

Conclude:

$$\Longrightarrow \boxed{\frac{1}{\#(G)} \sum_{t\in G} \rho_{\nu\mu}(t)\cdot \rho_{ij}(t^{-1}) = \frac{1}{\dim(V)}\,\delta_{i\mu}\,\delta_{\nu j}}$$

Case when $k = \mathbb{C}$ (Usually used)

May assume: ($\because$ Can give V, W hermitian inner product structure, and may Assume these inner product and the action of G are compatible. — by averaging a given inner product)

$(r_{\mu\nu})$, (ρ_{ij}) are unitary

i.e. $\rho_{ij}(t^{-1}) = \overline{\rho_{ji}(t)}$

Then can restate the orthogonality relation (of unitary matrix repr.) as

1) $V \not\cong W \Rightarrow \int_{t\in G} r_{\mu\nu}(t)\, \overline{\rho_{ij}(t)}\,dt = 0 \quad \forall \mu, \nu, i, j$

2) $V = W \Rightarrow \int_{t\in G} \rho_{ij}(t) \cdot \overline{\rho_{ij}(t)}\,dt = \frac{1}{\dim(V)} \cdot \delta_{ij}$

and $\int_{t\in G} \rho_{ij}(t)\, \overline{\rho_{k\ell}(t)}\,dt$

$= 0$ if either $i \neq k$ or $j \neq \ell$

What are intrinsic? Trace

(ρ, V): finite dim^{l} repr. of G

$$\rightsquigarrow \chi_V : G \longrightarrow k$$
$$t \longmapsto \mathrm{Tr}_V(\underbrace{\rho(t)}_{\in GL(V)})$$

Consequence of the orthogonality relation:

V, W: irreducible

$$\boxed{\int_{t \in G} \chi_V(t) \cdot \chi_W(t^{-1}) = \begin{cases} 0 & V \underset{k[G]}{\not\cong} W \\ 1 & V \underset{k[G]}{\cong} W \end{cases}}$$

$$\frac{1}{\#(G)} \sum_{t \in G} \chi_V(t) \underbrace{\chi_W(t^{-1})}_{\overline{\chi_W(t)}} \quad \text{if } k = \mathbb{C}$$

$$= \begin{cases} 0 & V \not\cong W \\ 1 & V \cong W. \end{cases}$$

How to think about this (so far), admitting a few facts which will be explained.

Assume $k = \mathbb{C}$ $\quad$ G: finite

1) G has only a finite nber of irred. repr up to isomorphism

($\therefore$ # of irred characters $\leq \#(G)$)

2) An irred. repr of G is determined by its character, up to isomorphism.

3) (Irreducible) characters are class functions on G (= functions on G depending only on conjugacy classes)

($\Rightarrow$ # irred repr $\leq$ # of conjugacy classes of G)

Fact
$\underline{\underline{*}}$ 4) # irrd repr $=$ # of conjugacy classes of G $= r$

Fact
$*$ 5) $\mathbb{C}[G] \cong \sum_{i=1}^{r} n_i \underbrace{(\rho_i, V_i)}_{\text{irred. rep}}$ $\Rightarrow \chi_{\mathbb{C}[G]} = \sum_i n_i \chi_{\rho_i}$

$\mathbb{C}[G]$ regarded as a left module over $\mathbb{C}[G]$

$\Rightarrow n_i = \dim(V_i) = \underbrace{\chi_{\rho_i}}_{\text{character of } \rho_i}(1_G)$

To see this:

$$(\chi_{\rho_i} \mid \chi_{\rho_i}) = 1$$

$$(\chi_{\rho_j} \mid \chi_{\rho_i}) = 0 \quad \text{if } i \neq j$$

$$\Longrightarrow n_i = (\chi_{\mathbb{C}[G]} \mid \chi_{\rho_i})$$

$$\Rightarrow \sum_{i=1}^{r} n_i^2 = (\chi_{\mathbb{C}[G]} \mid \chi_{\mathbb{C}[G]}) = \frac{1}{\#(G)} \cdot \#(G) \cdot \#(G) = \#(G)$$

To compute $\chi_{\mathbb{C}[G]}$: Use the $[t]$ $t \in G$ as a basis of $\mathbb{C}[G]$
$\Rightarrow$ $\forall t \in G$, t acts on $\mathbb{C}[G]$ through a permutation matrix, and all diagonal elements are 0 if $t \neq 1_G$

$$\Rightarrow \chi_{\mathbb{C}[G]}(t) = \begin{cases} \#(G) & \text{if } t = 1_G \\ 0 & \text{if } t \neq 1_G \end{cases}$$

$$\Rightarrow \underset{\overset{\|}{n_i}}{\left(\chi_{\mathbb{C}[G]} \,\middle|\, \chi_{\rho_i}\right)} = \frac{1}{\#(G)} \sum_{t \in G} \chi_{\mathbb{C}[G]}(t) \overline{\chi_{\rho_i}(t)}$$

$$= \frac{1}{\#(G)} \underbrace{\chi_{\mathbb{C}[G]}(1_G)}_{\#(G)} \cdot \underbrace{\overline{\chi_{\rho_i}(1_G)}}_{\dim(V_i)}$$

$$= \dim(V_i) = \chi_{\rho_i}(1)$$

Conclusion:

Def^n: The character table of G is the following $r \times r$ matrix $\quad r = \#$ conj. classes $= \#$ irred repr.
(after choosing an order of irred. repr and an order of conjugacy classes.)

Say $1 = \chi_1, \dots, \chi_r$ are the r irred characters

$\{1_G\} = C_1, \dots, C_r$ are the r conjugacy classes

Let $c_i = \# C_i$

Character table.

a list of values of irred. char.

	$\{1_G\} = C_1$ (1, c_1)	C_2 (c_2)	$\cdots$ C_j (c_j) $\cdots$	C_r (c_r)
$1 = \chi_1$	1	1	1	1
χ_2	$\chi_2(1)$			
$\vdots$				
χ_μ	$\chi_1(1)$		$\chi_\mu(t_j)$, $t_j \in C_j$	
$\vdots$				
χ_r	$\chi_r(1)$			

Orthogonality relations

$$\boxed{1} \quad \frac{1}{\#(G)} \sum_{j=1}^{r} c_j \chi_\mu(t_j) \cdot \overline{\chi_\nu(t_j)} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu \end{cases}$$

$\Updownarrow$

$$\boxed{2} \quad \frac{c_j}{\#(G)} \sum_{\mu=1}^{r} \chi_\mu(t_j) \overline{\chi_\mu(t_k)} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Ex 1. $\mathbb{Z}/n\mathbb{Z}$

Pick a primitive n-th root of 1, $\zeta = \zeta_n$

Define $\forall i \in \mathbb{Z}/n\mathbb{Z}$, a 1-dim^{l} repr / character

$$\chi_i : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$$

by $\quad \chi_{\bar{i}}(\bar{m}) \overset{\text{def}}{=} \zeta^{m}$

$\bar{m} = m + n\mathbb{Z}$

and $\quad \chi_{\bar{0}}, \chi_{\bar{1}}, \dots, \chi_{\overline{n-1}}$

$\chi_{\bar{0}} = \mathbb{1}$

	$\bar{0}$	$\bar{1}$	$\dots$	$\bar{j}$	$\dots$	$\overline{n-1}$
$\chi_{\bar{0}}$						
$\chi_{\bar{1}}$						
$\vdots$						
$\chi_{\bar{i}}$				$\zeta^{\bar{i}\cdot\bar{j}}$		
$\chi_{\overline{n-1}}$						

<u>Ex.</u> S_3 conjugacy class: $\underset{C_1}{\{1\}}$, $\underset{C_2}{\{(12),(13),(23)\}}$, $\underset{C_3}{\{(123),(132)\}}$

	$\overset{1}{C_1}$	$\overset{3}{C_2}$	$\overset{2}{C_3}$
$\mathbb{1} = \chi_1$	1	1	1
$\chi_{sgn} = \chi_2$	1	-1	1
χ_3	2	0	-1

$\longleftarrow$ action of $S_3 = D_6$ on $\mathbb{R}^2$ (and $\mathbb{C}^2$)

$S_3 = D_6$ dihedral group with 6 elts.

Has a 2-dim^{l} repr. $/\mathbb{R}$, V

$V \otimes_{\mathbb{R}} \mathbb{C}$: is a 2-dim^{l} repr.

The eigenvalues of this repr

- reflections : 1 & -1 $\rightsquigarrow$ $tr = 0$
- rotations by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ $\quad e^{\frac{2\pi\sqrt{-1}}{3}}, e^{\frac{-2\pi\sqrt{-1}}{3}}$ $\rightsquigarrow$ $tr = -1$

So this repr$/\mathbb{C}$ has character $= \chi_3$

Fact
<u>Lemma</u>: If a $\mathbb{C}$-repr. (V, ρ) satisfies

$$(\chi_V | \chi_V) = 1,$$

<u>then</u> (V, ρ) is irred.

<u>why</u> $\quad V \cong \sum_{i=1}^{r} m_i V_i \qquad m_i \in \mathbb{N}$

$$\underset{\substack{\parallel \\ 1}}{(\chi_V | \chi_V)} = \sum_{i=1}^{r} m_i^2$$

O.K.