Linear representation of finite groups

ref: Serre, Linear representations of finite group

- Curtis and Reiner, Representation theory of finite groups and associative algebras.
  - * Very useful
  - * Gives insight into semisimple algebras. (ring)

Basics

Def^n: A linear representation of a group G (on a vector space) over a field k is

equiv.

a) a group homomorphism $G \to GL_k(V)$
V = a vector space over k

b) a ring homomom

$$k[G] \longrightarrow \mathrm{End}_k(V)$$

V: a vector space over k

c) a left $k[G]$-module structure (on a vector space V over k)

Reason for concentrating on groups G:
can encode information effectively in terms of functions on G
(central/class)

Will deal (most of the time) with the case when G is finite and the order of G is invertible in k. i.e. either $\mathrm{char}(k)=0$ or $\mathrm{char}(k) \nmid \#(G)$.

$$\mathrm{char}(k) = p > 0$$

Lemma: Assume either G is finite and $\#(G)\cdot 1 \in k^\times$

or G is a compact Hausdorff topological group and $k = \mathbb{R}$ or $\mathbb{C}$.

Then every short exact sequence

$$0 \to V' \xrightarrow{i} V \xrightarrow{\pi} V'' \to 0$$

of finite dim'l linear representations of G splits,

i.e. $\exists$ a k-linear map $\varepsilon : V'' \to V$

s.t. ε is G-equivariant

and $\pi \circ \varepsilon = \mathrm{Id}_{V''}$.

Pf: Uses: average / integral over G

- W is a left G-module

$\rightsquigarrow$ $\int_G : W \longrightarrow W$

$$w \longmapsto \int_{y \in G} y\cdot w \, dy = \begin{cases} \frac{1}{\#(G)} \sum_{y \in G} y \cdot w & G \text{ finite } \#(G)\cdot 1 \in k^\times \\ \text{integral of } y\cdot w \text{ for the Haar measure of } G \text{ normalized by } \int_G 1\, dy = 1 \end{cases}$$

Fact: Let G be a locally compact Hausdorff topological group.

1) There exists a (left) Haar measure on G

i.e. a non-trivial Borel measure μ on G

s.t. $\mu(x \cdot S) = \mu(S)$

$\forall$ Borel subset $S \subseteq G$

$\forall\ x \in G$

2) Any two left Haar measures on G diff by a constant $\in \mathbb{R}^{\times}_{>0}$.

Note: If G is compact, then every left Haar measure is a right Haar measure.

Usually: normalize by $\displaystyle\int_G 1 \cdot d\mu = 1$

Exer. Determine the Haar measures on

(a) $GL_n(\mathbb{R})$

(b) $GL_n(\mathbb{C})$

compact

(c) $O_n(\mathbb{R})$

(d) $SO_n(\mathbb{R})$

(e) $U_n(\mathbb{C})$

(f) $SU_n(\mathbb{C})$

E.g. $\mathbb{R}^{\times}$: $\dfrac{dx}{|x|}$

$\mathbb{C}^{\times}$: $\dfrac{|dz \wedge d\bar{z}|}{z \cdot \bar{z}}$

Given $\quad 0 \to V' \xrightarrow{i} V \xrightarrow[\text{G-equiv.}]{\pi} V'' \to 0$ short exact seq of G-mod.

Pick $\quad h : V'' \to V$
$\quad h \in \operatorname{Hom}_k(V'', V)$ s.t. $\pi \circ h = \mathrm{id}_{V''}$.

Let $\quad \varepsilon = \int_G h = \frac{1}{\#(G)} \sum_{x \in G} (x \cdot h)$

exer. $\pi \circ (x \cdot h) = \mathrm{Id}_{V''}$ $\quad$ where $\quad (x \cdot h)(v'') = x(h(x^{-1} v''))$ $\quad \forall v'' \in V''$

Note: $\operatorname{Hom}_k(V'', V)$ has a natural left G-module structure:

$$\underset{\substack{\uparrow \\ G}}{(x} \cdot h)\,(\underset{\substack{\uparrow \\ V''}}{v''}) = \underbrace{x \cdot (h(x^{-1} \cdot v''))}_{\in V}$$

exer a) $\pi \circ \varepsilon = \mathrm{Id}_{V''}$

b) $\varepsilon = \int_G x \cdot h$ has the property that $y \cdot \varepsilon = \varepsilon$ $\forall y \in G$

i.e. ε is G-equivariant.

q.e.d.

Immediate consequences $\quad$ Assume one of the cond^{n}s hold

Every finite dim^{l} G-modules is isomorphic to a finite direct sum of irreducible (simple) G-modules!

<u>Schur's Lemma</u>: Assume k is algebraically closed.

V: finite dim'l irred repr. of G

<u>Thm</u>: $\mathrm{End}_G(V) = k$.

<u>Pf</u>: V is irred $\underset{\text{(exer.)}}{\Longrightarrow}$ $\mathrm{End}_G(V)$ is a division alg. $/k$

$\mathrm{End}_G(V) \ni \alpha$

V finite dim'l + k alg. closed

$\Longrightarrow \quad \exists\, \lambda \in k$ s.t. $\underbrace{\mathrm{Ker}(\alpha - \lambda\cdot \mathrm{Id}_V)}_{\text{stable under } G} \neq (0)$

$\Longrightarrow \quad \mathrm{Ker}(\alpha - \lambda\cdot \mathrm{Id}_V) = V$

<u>i.e.</u> $\alpha = \lambda \cdot \mathrm{Id}_V$. q.e.d.

<u>Consequences</u>: Assume k is algebraically closed

Let V and W be two <u>irred</u> repr. of G

$$\underbrace{\int_G}_{\frac{1}{\#G}\sum_{x\in G} x\cdot(?)} : \underbrace{\mathrm{Hom}_k(V, W)}_{} \longrightarrow \underbrace{\mathrm{Hom}_{k[G]}(V, W)}_{} $$

a vector space $\triangleq$ a space consisting of <u>matrices</u>

$$\mathrm{Hom}_{k[G]}(V,W) = \begin{cases} 0 & V \underset{G}{\not\cong} W \\ k\cdot \mathrm{Id}_V & \text{If } V = W \end{cases}$$

Pick a basis $\begin{cases} v_1, \dots, v_n & \text{of } V \\ w_1, \dots, w_m & \text{of } W \end{cases}$

The action of G on V is given by

$$G \ni x \longmapsto \underbrace{\left(\rho_{ij}(x)\right)_{1 \le i, j \le n}}_{\text{matrix repr. for } V}$$

Similarly, the action of G on W is given by

$$G \ni y \longmapsto \left(r_{\mu\nu}(y)\right)_{1 \le \mu, \nu \le n}$$

Action of G on $\underbrace{\mathrm{Hom}_k(V, W)}_{\Updownarrow}$

$m \times n$ matrices

$$G \ni x : \quad (z_{\nu i}) \longmapsto r_{\mu\nu}(x) \cdot z_{\mu i} \cdot \rho_{ij}(x^{-1})$$

The map $\int_G$ is

$$(*) \quad (z_{\nu i}) \longmapsto \frac{1}{\# G} \sum_{x \in G} \sum_{\substack{\nu, i \\ \nu \in 1, \dots, m \\ i \in 1, \dots, n}} \overbrace{r_{\mu\nu}(x)}^{\text{change to } \rho_{\mu\nu}} \cdot \underline{z_{\nu i}} \cdot \rho_{ij}(x^{-1}) \quad !$$

Assume $V \underset{G}{\not\cong} W \Rightarrow$ the map given by $(*)$ is 0

This is an equation in the variables $z_{\nu i}$

So, all of its coeff are 0.

$$\Longrightarrow \quad \sum_{x \in G} r_{\mu\nu}(x) \cdot \rho_{ij}(x^{-1}) = 0$$

$$\forall \mu, \nu, \ \forall i j.$$

Exercise: $V = W$

Then use the same basis for $V = W$

Schur's Lemma

$$\operatorname{Hom}_k(V, V) \ni h \longmapsto \int_G h = \underbrace{c}_{\overset{\|}{?}} \cdot \mathrm{Id}_V$$

$$\operatorname{Tr}\left(\int_G h\right) = \operatorname{Tr}(h)$$

$$\Rightarrow c = \frac{\operatorname{Tr}(h)}{\#(G)}$$

Again, have an equation in variables $z_{\nu i}$

Find this equation Explicitly!!