Associated primes, primary decomposition

A : a commutative ring. <u>Assume</u> A is <u>Noetherian</u>

<u>Def</u> Let M be an A-module.

$$\mathrm{Ass}_A(M) = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \exists\, x \in M \ (x \neq 0) \text{ s.t. } \mathfrak{p} = \underbrace{\mathrm{Ann}_A(x)}_{\overset{\mathrm{def}}{=} \{ y \in A \mid xy = 0 \}} \}$$

↑ the set of associated primes of M

<u>Lemma</u>: Every maximal elt of the family $\{ \mathrm{Ann}_A(x) \mid x \in M,\ x \neq 0 \} =: \mathcal{F}$ is a prime ideal of A.

<u>Pf</u>: Let $\mathfrak{P}$ be a maximal elt in $\mathcal{F}$. ($\mathfrak{P} \neq A$)

Suppose $a, b \in A$, $a \cdot b \in \mathfrak{P}$, $a \notin \mathfrak{P} = \mathrm{Ann}_A(x_0)$, $x_0 \in M$

$\Updownarrow$ $a \cdot x_0 \neq 0$

$$\mathrm{Ann}_A(a x_0) \supseteq \mathrm{Ann}_A(x_0) = \mathfrak{P}$$

$$\Longrightarrow \mathrm{Ann}_A(a x_0) = \mathrm{Ann}_A(x_0) = \mathfrak{P}$$

$b \in \mathrm{Ann}_A(a x_0)$

q.e.d.

<u>Cor</u>: $M \neq (0) \iff \mathrm{Ass}_A(M) \neq \phi$

<u>Cor</u>. $a \in A$.

$[a]_M : M \to M$ is injective $\iff a \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathfrak{p}$

"$\Leftarrow$" Suppose $a \in \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathfrak{p}$

$\forall x \in M$ ($x \neq 0$), if $ax = 0$ $\Longrightarrow$ $\exists\, \mathfrak{p} \in \mathrm{Ass}_A(M) \ni a$, $\mathfrak{p} \supseteq \mathrm{Ann}(x)$ <u>contradiction</u>

"$\Rightarrow$" <u>Exer</u>.

[Will see shortly: $|\mathrm{Ass}_A(M)| < \infty$]

Equiv. statement of the last Cor:

$$\bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathfrak{p} = \text{the set of all zero divisors for } M$$

$$= \{ b\in A \mid \exists\, x\in M,\ x\neq 0 \text{ s.t. } b\cdot x = 0 \}$$

To show that $\mathrm{Ass}_A(M)$ is finite

__Lemma__: $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ short exact sequence of A-module.

$$\Longrightarrow \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(M') \cup \mathrm{Ass}_A(M'')$$

__Pf__: __Suppose__ $\mathfrak{p} \notin \mathrm{Ass}_A(M') \cup \mathrm{Ass}_A(M'')$

and $\mathfrak{p} \in \mathrm{Ass}_A(M)$ Want a __contradiction__.

$\exists\, y_0 \in M$ st $\mathfrak{p} = \mathrm{Ann}(y_0)$

- If $\beta(y_0) \neq 0 \Longrightarrow \exists\, \mathfrak{q} \in \mathrm{Ass}_A(M'')$ __s.t.__ $\mathfrak{q} \supseteq \mathrm{Ann}(\beta(y_0)) \supseteq \mathfrak{p}$

- If $\beta(y_0) = 0 \Longrightarrow y_0 \in M'$ __and__ $\mathfrak{p} = \mathrm{Ann}_A(y_0) \in \mathrm{Ass}_A(M')$

<u>Conclusion</u> : $\mathfrak{q} \cancel{\subseteq \bigcup_{Q' \in \mathrm{Ann}(M')}} Q' \cup \bigcup_{Q'' \in \mathrm{Ann}(M'')} Q''$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
 & & A/\mathfrak{q} \arrow[u, phantom, "\subseteq" rotate=90] & & \\
 & & N \arrow[u, phantom, "\wr\!\!\!\!\!\parallel"] & &
\end{tikzcd}$$

$$\mathfrak{q} \in \mathrm{Ass}_A(M) \iff \exists\, x_0 \in M,\ x_0 \neq 0 \text{ s.t. } \mathrm{Ann}_A(x_0) = \mathfrak{q}$$

$$\iff \exists\, N \subseteq M \text{ A-submodule}, \quad N \cong A/\mathfrak{q}$$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, hook] & M \arrow[r] & M'' \arrow[r] & 0. \\
 & & N \arrow[u, phantom, "\subseteq" rotate=90] & &
\end{tikzcd}$$

- If $N \cap M' \neq (0) \Rightarrow \forall\, y_1 \neq 0,\ y_1 \in N \cap M'$, $\mathrm{Ann}_A(y_1) = \mathfrak{q} \in \mathrm{Ass}_A(M')$

  $N \cong A/\mathfrak{q}$ is an integral domain.

- If $N \cap M' = (0)$

  $\Rightarrow$ The map $M \twoheadrightarrow M''$ induces an A-linear embedding

  $$A/\mathfrak{q} = N \hookrightarrow M''$$

  $\Rightarrow \mathfrak{q} \in \mathrm{Ass}_A(M'')$. <u>q.e.d.</u>

$$\mathfrak{p} \in \mathrm{Ass}_A(M) \iff \exists \text{ a submodule of } M \text{ isomorphic to } A/\mathfrak{p}$$

<u>Cor</u>: If M is a finite A-module, <u>then</u> $|\mathrm{Ass}_A(M)| < \infty$.

Pf: $M = (0)$ OK/trivial

$M \neq 0 \Rightarrow \exists\, N \subseteq M$, $N \cong A/\mathfrak{p}$, $\mathfrak{p} \in \mathrm{Spec}(A)$

$\rightsquigarrow N \subsetneqq M$, $N \cong A/\mathfrak{p}$, $\mathfrak{p} \in (A)$

Noetherian induction:

$\exists$ a finite sequence of A-submodules

$$(0) = M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots \subsetneqq M_r = M$$

<u>s.t.</u> $M_{i+1}/M_i \cong A/\mathfrak{p}_i$, $\mathfrak{p}_i \in \mathrm{Spec}(A)$ for $i = 0, \dots, r-1$

$$\Rightarrow \mathrm{Ass}_A(M) \subseteq \bigcup_{0 \le i \le r-1} \underbrace{\mathrm{Ass}(M_{i+1}/M_i)}_{=\{\mathfrak{p}_i\}}$$

$$= \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$$

<u>q.e.d.</u>

Picture: M: a finite A-module.

Consider $\widetilde{M}$: the sheaf of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules associated to M

$$\mathrm{Support}(\widetilde{M}) \overset{?}{=}$$

$$\| $$

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \widetilde{M}_{\mathfrak{p}} \neq (0)\}$$

exer: is a closed subset of $\mathrm{Spec}(A)$

$\widetilde{M}_{\mathfrak{p}}$ ← the stalk of $\widetilde{M}$ at the point $\mathfrak{p} \in \mathrm{Spec}(A)$

$M_{\mathfrak{p}}$ ← localization of M w.r.t $(A \setminus \mathfrak{p})$

Phenomenon:

$$\mathrm{supp}(\widetilde{M}) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \underbrace{\mathrm{Spec}(A/\mathfrak{p})}_{\substack{\| \\ V(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \supseteq \mathfrak{p}\}}}$$

Example: $\underbrace{\mathbb{C}[x,y]}_{\substack{\| \\ A}} \Big/ \underbrace{(x(y^3-x))}_{\substack{\| \\ I}} = M$

What is $\mathrm{Ass}_A(M) \overset{?}{=} \{(x), (y^3-x)\}$

"$\supseteq$" $\quad \bar{x} := x + I \in A/I = M \qquad \underbrace{(y^3-x)+I}_{\bar{x}''}$

$$\mathrm{Ann}_A(\bar{x}) = (y^3-x)$$

$$\mathrm{Ann}_A(\bar{x}'') = (x)$$

$$\mathbb{C}[x,y]/(x(y^3-x)) \xrightarrow[\text{canonical}]{\gamma} \mathbb{C}[x,y]/(x) \oplus \mathbb{C}[x,y]/(y^3-x)$$

a <u>UFD</u>

$$\overline{f(x,y)} \longmapsto (0,\ 0)$$

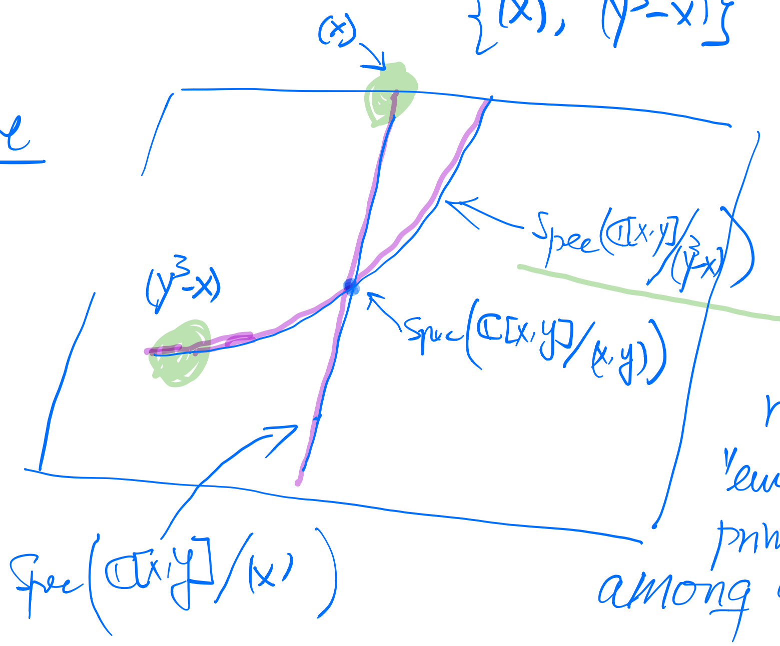
$x,\ y^3-x$ are irred $\Longrightarrow$ $f(x,y) \in x\cdot\mathbb{C}[x,y]$

$f(x,y) \in (y^3-x)\cdot\mathbb{C}[x,y]$

$\Longrightarrow f(x,y) \in x\cdot(y^3-x)\cdot\mathbb{C}[x,y]$

$$\Longrightarrow \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A\left[\left(\mathbb{C}[x,y]/x\right) \oplus \left(\mathbb{C}[x,y]/(y^3-x)\right)\right] = \{(x),\ (y^3-x)\}$$

<u>Picture</u>



<u>may have</u> "embedded prime ideals" among associated primes.

$$\underline{\underline{\mathrm{Supp}(M) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathrm{Spec}(A/\mathfrak{p})}}$$