Primary decomposition of modules

Have defined: associated primes

A: Noetherian, commutative.

Prop Let M be a finite A-module. $\Phi \subseteq \mathrm{Ass}_A(M)$

$\exists$ an A-submodule $N \subseteq M$ such that

$$\begin{cases} \mathrm{Ass}_A(N) = \mathrm{Ass}_A(M) \setminus \Phi \\ \mathrm{Ass}_A(M/N) = \Phi \end{cases}$$

Pf: strategy: Consider the family $\mathcal{F}$ of all A-submodule $N \subseteq M$

s.t. $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M)$

Let N_1 be a maximal elt in $\mathcal{F}$.

Want to show:

$$\mathrm{Ass}_A(M/N_1) = \Phi. \quad \left(\text{and} \quad \mathrm{Ass}_A(N_1) = \mathrm{Ass}_A(M) \setminus \Phi\right)$$

$$\therefore \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N)$$

Suppose $\mathrm{Ass}_A(M/N_1) \ni \mathfrak{p}$, $\mathfrak{p} \notin \Phi$

$\rightsquigarrow$ $\exists$ a submodule $Q \subseteq M/N_1$, $Q \cong A/\mathfrak{p}$.

$\Rightarrow$ ~~$N_1 \cap Q = (0)$~~

Consider ~~$N_1 + Q \subseteq M$~~ $N_1 \subseteq \tilde{N}_1 \subseteq M$ $\quad \tilde{N}_1/N_1 = Q$

$$\mathrm{Ass}_A(\widetilde{N_1 \cancel{+ Q}}) \subseteq \mathrm{Ass}_A(N_1) \cup \mathrm{Ass}_A(Q)$$
$$\subseteq \mathrm{Ass}_A(M) - \Phi$$

a contradiction!

QED.

Lemma / Exer.

S: multiplicative subset of $A \setminus \{0\}$

(a) $\mathrm{Ass}_A(M) \cap \mathrm{Spec}(S^{-1}A) \leftrightarrow \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$

[True with assuming any noeth. cond on A or M]

(b) A Noeth. $\mathfrak{p} \in \mathrm{Spec}(A)$.

$S^{-1}\mathfrak{p} \, (= \mathfrak{p} \cdot S^{-1}A) \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$

then $\mathfrak{p} \in \mathrm{Ass}_A(M)$

(Compatibility of the formation of $\mathrm{Ass}_A(M)$ with localization)

Key Prop A Noeth. M: A-module $S \subseteq A \setminus \{0\}$ multiplicative

$\Psi = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \cap S = \phi \}$

Exer. Then $\exists!$ a submodule $N \subseteq M$

s.t. $\mathrm{Ass}_A(N) = \mathrm{Ass}_A(M) \setminus \Psi$

and $\mathrm{Ass}(M/N) = \Psi$.

Def^n. A: ring. M: A-module

Say M is <u>co-primary</u> if $\forall a \in A$

$[a]_M$ is either injective or <u>almost nilpotent</u>

in the sense that $\forall x \in M$, $\exists n(a,x) \in \mathbb{N}$ s.t. $a^{n(a,x)} \cdot x = 0$.

[Note: No need for the notion of "almost nilp" when M is a finite A-module.

Lemma: Suppose A is Noetherian. Then an A-module M is co-primary iff $\mathrm{Ass}_A(M)$ is a singleton $\{\mathfrak{p}\}$.

(Exer.)

Note: Let Q be an ideal of A

Q is primary $\Longleftrightarrow$ A/Q is a coprimary A-module

Theorem: Assume M is a finite A-module, A: Noetherian

$N \subseteq M$

Then: $\exists$ a finite family of A-submodules $N \subseteq Q_i \subseteq M$ s.t. M/Q_i is co-primary $\forall i$, and $\bigcap_i Q_i = (0)$

Pf: Apply key proposition to

$\overline{M} = M/N$ (i.e. may assume $N=0$)

and for each $\mathfrak{p}_i \in \mathrm{Ass}_A(\overline{M})$

produce a submodule $\overline{N}_i \subseteq \overline{M}$

s.t. $\mathrm{Ass}_A(\overline{N}_i) = \mathrm{Ass}_A(\overline{M}) \setminus \{\mathfrak{p}_i\}$.

$$\left(\Rightarrow \bigcap_i \mathrm{Ass}(\overline{N}_i) = \phi \right.$$
$$\cup\!\!|$$
$$\mathrm{Ass}\left(\bigcap_i \overline{N}_i\right)$$

q.e.d.

Case $M = A$, $N = (0)$

This says: $\exists$ primary ideals $Q_1, \cdots, Q_m$

s.t. $Q_1 \cap \cdots \cap Q_m = (0)$

Equivalent Formation:

$\forall$ ideal $I \subseteq A$.

$\exists$ primary ideals $Q_1, \cdots, Q_m$ in A

s.t. $Q_1 \cap \cdots \cap Q_m = I$.

$$\Rightarrow \quad \mathrm{Spec}(A) = \bigcup_{i=1}^{m} \mathrm{Spec}(A/Q_i)$$

$$\left(\Rightarrow \quad A/\mathrm{rad}(I) = P_1 \cap \cdots \cap P_m \right)$$

where P_i = the prime ideal of A associated to the primary ideal Q_i

More explicitly: The last statement says that $\forall \mathfrak{p} \in \mathrm{Spec}(A)$

$$\mathfrak{p} \supseteq I \iff \mathfrak{p} \supseteq P_i \text{ for some } i=1,\dots,m.$$

Theorem $\underset{M}{A} \longrightarrow \underset{N}{B}$

A, B: commutative.
M: A-module
N: B-module

Assume: N is flat over A

Then 1) $\mathrm{Ass}_B(M \otimes_A N) \supseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_A(M)} \mathrm{Ass}_B(N/\mathfrak{p}N)$

2) A: Noethen M: f.g A-module

$$\boxed{\mathrm{Ass}_B(M \otimes_A N) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathrm{Ass}_{B/\mathfrak{p}}(N/\mathfrak{p}N)}$$

Note In application

$$\begin{array}{ccc} \mathrm{Spec}(B) & N, & M \otimes_A N \\ \text{flat} \big\downarrow & & \\ \mathrm{Spec}(A) & M & \end{array}$$

1): Given $\mathfrak{p} \in \mathrm{Ass}_A(M)$

i.e $A/\mathfrak{p} \hookrightarrow M$

$\overset{\rightsquigarrow}{N \text{ is } A\text{-flat}}$ $N/\mathfrak{p}N \hookrightarrow M \otimes_A N$ B-linear

$\Rightarrow \mathrm{Ass}_B(N/\mathfrak{p}N) \subseteq \mathrm{Ass}_B(M \otimes_A N)$

Lemma: M': an A-module f.g. s.t. $\mathrm{Ass}_A(M') = \{\mathfrak{q}\}$
N: A-flat

Then $\forall \mathfrak{P} \in \mathrm{Ass}_B(M' \otimes_A N)$

have $\mathfrak{P} \cap A = \mathfrak{q}$

Pf: Choose $n \in \mathbb{N}$ s.t. $\mathfrak{q}^n M' = 0$.

Given $\mathfrak{P} \in \mathrm{Ass}_B(M' \otimes_A N)$,

$\rightsquigarrow \quad B/\mathfrak{P} \xhookrightarrow[B\text{-linear}]{} M' \otimes_A N$

$\Longrightarrow \mathfrak{q}^n B \subseteq \mathfrak{P}$

Also: elements of $A \setminus \mathfrak{q}$ operates as injections on M'

$\rightsquigarrow \quad \mathfrak{P} \cap A = \mathfrak{q}$. q.e.d.

(2) Assume first that $\mathrm{Ass}_A(M) = \{\mathfrak{q}\}$.

Lemma $\Rightarrow$ $\forall \mathfrak{P} \in \mathrm{Ass}_B(M \otimes_A N)$
have $\mathfrak{P} \cap A = \mathfrak{q}$

Have a filtration

$(0) \subseteq M_1 \subsetneq \cdots \subsetneq M_m = M$ st $M_i/M_{i+1} \cong A/\mathfrak{q}_i$ for all i

$\otimes$ the filtration with N

$\rightsquigarrow \quad \mathrm{Ass}_B(M \otimes_A N)$

$\subseteq \bigcup_{i=1}^{n} \mathrm{Ass}_B(N/\mathfrak{q}_i N) \cap {}^a\varphi^{-1}(\{\mathfrak{q}\})$

${}^a\varphi: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$

$$\underset{\text{Lemma}}{\Longrightarrow} \quad \mathrm{Ass}_B(M \otimes N) \subseteq \mathrm{Ass}_B(N/\mathfrak{p}N)$$

<u>General case</u>:

Use primary decomposition of M:

$$\underset{\underset{M}{\cap}}{(0)} = Q_1 \cap \cdots \cap Q_n$$

$$\mathrm{Ass}_A(M/Q_i) = \{\mathfrak{p}_i\} \quad \forall i = 1, \dots, n$$

$$\mathrm{Ass}_B(M \otimes_A N) \subseteq \bigcup_i \mathrm{Ass}_B((M/Q_i) \otimes_A N)$$

$$= \bigcup_i \mathrm{Ass}_B(N/\mathfrak{p}_i N).$$

<u>q.e.d</u>

<u>Geometrically</u> (i.e. ignoring nilpotents)

$$A = \mathbb{C}[x_1, \dots, x_n] / I \leftarrow I = (f_1, \dots, f_m)$$

$$I = Q_1 \cap \cdots \cap Q_n$$

Q_i: P_i-primary

P_i = prime

$P_1, \dots, P_n$: distinct.

$$\Downarrow$$

$$\mathrm{rad}(I) = P_1 \cap \cdots \cap P_n$$

$$\mathrm{rad}(I) = \{x \in A \mid \exists n \in \mathbb{N} \text{ s.t. } x^n \in I\}$$

$$\mathrm{Ass}_A(A) = \{P_1, \dots, P_n\}$$

It may happen that $P_i \subsetneq P_j$ for some i, j.

those P_j containing a P_i, $i \neq j$ are called "embedded primes"

$$\mathrm{rad}(I) = \bigcap_{\substack{P \in \mathrm{Ass}_A(A) \\ \text{non-embedded}}} P$$

$I \subseteq \mathrm{rad}(I)$

$\exists\, n_0 \in \mathbb{N}$ s.t. $\mathrm{rad}(I)^{n_0} \subseteq I$

$$\mathrm{Spec}(A/I) = \mathrm{Spec}\left(A/\mathrm{rad}(I)\right)$$

$$= \bigcup_{P:\,\text{non-embedded}} \mathrm{Spec}(A/P).$$

each such $\mathrm{Spec}(A/P)$ is an irreducible component of $\mathrm{Spec}(A/I)$