Toward Tor and Hom
(homological algebra)
– projective / injective module

<u>Def</u> Let R be a ring, and let M be a left R-module

(1) M is a <u>projective</u> left R-module iff

$$\mathrm{Hom}_R(M, ?) : \begin{pmatrix}\text{left } R\text{-modules}\end{pmatrix} \to \begin{pmatrix}\text{abel.} \\ \text{groups}\end{pmatrix}$$
$$N \longmapsto \mathrm{Hom}_R(M, N)$$

is <u>exact</u>.

<u>E.g.</u> Every <u>free left R-module</u> is projective.

$$\Big(\cong \bigoplus_{i \in I} R \Big\}$$

(2) M is an <u>injective</u> left R-module if

$$\mathrm{Hom}_R(?, M) : \begin{pmatrix}\text{left } R\text{-modules}\end{pmatrix} \to \begin{pmatrix}\text{abelian} \\ \text{groups}\end{pmatrix}$$

is exact.

<u>E.g.</u> (a) If R is a division ring, then every R-module is both injective and projective. (Exer.)

(b) <u>$R = \mathbb{Z}$</u> Every $\mathbb{Q}$-vector space (module), regarded as a $\mathbb{Z}$-module is injective

Exer A $\mathbb{Z}$-module M is injective iff M is <u>divisible</u>

$M \xrightarrow{[n]} M$ (onto) $\forall\, 0 \neq n \in \mathbb{Z}$

"only iff": easier.

e.g. $\mathbb{Q}/\mathbb{Z}$ is divisible

$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \gcd(b,p)=1 \right\}$

a prime number

$\mathbb{Q}/\mathbb{Z}_{(p)}$

$\mathrm{Hom}_R(\cdot\,, M)$

$\mathrm{Hom}_R(M, \cdot\,)$ are both left exact (but generally not right exact)

E.g. Given $0 \to N' \to N \to N'' \to 0$ a short exact sequence of left R-modules. $\rightsquigarrow$ an exact sequence

$$0 \to \mathrm{Hom}_R(M,N') \xrightarrow{\alpha_0} \mathrm{Hom}_R(M,N) \xrightarrow{\beta_0} \mathrm{Hom}_R(N'')$$

Want info about this arrow

$$\xrightarrow{\delta^0} \mathrm{Ext}^1_R(M,N') \xrightarrow{\alpha_1} \mathrm{Ext}^1_R(M,N) \xrightarrow{\beta_1} \mathrm{Ext}^1_R(M,N'')$$

$$\xrightarrow{\delta^1} \mathrm{Ext}^2_R(M,N') \to \mathrm{Ext}^2_R(M,N) \to \mathrm{Ext}^2_R(M,N'')$$

$$\vdots$$

$$\to \mathrm{Ext}^i_R(M,N') \to \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N'')$$

$$\to \cdots \qquad i \in \mathbb{N}$$

$$\mathrm{Im}(\beta_0) = \mathrm{Ker}(\delta^0), \quad \dots, \quad \mathrm{Im}(\beta_i) = \mathrm{Ker}(\delta^i)$$

$$\mathrm{Im}(\delta_0) = \mathrm{Ker}(\alpha_1), \quad \dots$$

$$0 \to N' \to N \to N'' \to 0$$

$\rightsquigarrow$ long exact sequence

$$0 \to \mathrm{Hom}_R(N'', M) \xrightarrow{\alpha_0} \mathrm{Hom}_R(N, M) \xrightarrow{\beta_0} \mathrm{Hom}_R(N', M)$$
$$\xrightarrow{\delta^0} \mathrm{Ext}^1_R(N'', M) \to \mathrm{Ext}^1_R(N, M) \to \mathrm{Ext}^1_R(N', M)$$
$$\dots\dots$$
$$\to \mathrm{Ext}^i_R(N'', M) \to \mathrm{Ext}^i_R(N, M) \to \mathrm{Ext}^i_R(N', M)$$
$$\to \dots\dots$$

e.g. $\mathrm{Im}(\beta_0) = \mathrm{Ker}(\delta^0)$

$\rightsquigarrow$ have $\mathrm{coker}(\beta_0) \hookrightarrow \mathrm{Ext}^1_R(N'', M)$

How to get/define these Ext^i-groups?

Ans. To define $\mathrm{Ext}^i_R(M, N)$ — M, N: left R-module

3 ways

(1) Take a projective resolution of M

an exact sequence of projective left R-module indexed by $\mathbb{N}$.

$$\dots \to P_{i+1} \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} \dots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to 0$$

$\mathrm{Im}(\partial_0) = M$

$\mathrm{Ker}(\partial_0) = \mathrm{Im}(\partial_1)$

i = grading as chain complex

$-i$ = grading as a cochain complex

Look at

$$\mathrm{Hom}_R(P_\bullet, N) = \Big(0 \xrightarrow{\delta^{-1}} \mathrm{Hom}_R(P_0,N) \xrightarrow{\delta^0} \mathrm{Hom}_R(P_1,N) \xrightarrow{\delta^1} \cdots \to \cdots \xrightarrow{\delta^{i-1}} \underbrace{\mathrm{Hom}_R(P_i,N)}_{\deg = i} \xrightarrow{\delta^i} \mathrm{Hom}(P_{i+1},N) \to \cdots \Big)$$

Have $\delta^{i+1} \circ \delta^i = 0 \quad \forall i$. ← a cochain complex

$$H^i(\mathrm{Hom}_R(P_\bullet,N)) = \mathrm{Ext}^i_R(M,N) \underset{\text{def}}{:=} \mathrm{Ker}(\delta^i)/\mathrm{Im}(\delta^{i-1}) \quad \forall i$$

Fact Given any two projective resolutions

$P_\bullet \to M$ and $Q_\bullet \to M$

there exists a natural canonical isomorphism (depending on the two resolutions)

$$H^i(\mathrm{Hom}_R(P_\bullet,N)) \xrightarrow{\simeq} H^i(\mathrm{Hom}_R(Q_\bullet,N))$$

These canonical isomorphisms satisfy the natural $\begin{cases} \text{cocycle condition} \\ \text{associativity condition.} \end{cases}$

$P_\bullet \to M, \quad Q_\bullet \to M, \quad L_\bullet \to M$

$$\begin{tikzcd}
H^i(\mathrm{Hom}_R(P_\bullet,N)) \arrow[rr, "\simeq"'] \arrow[dr, "\simeq"'] & \arrow[d, phantom, "\text{commutes!}"] & H^i(\mathrm{Hom}_R(L_\bullet M)) \\
 & H^i(\mathrm{Hom}_R(Q_\bullet N)) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

$\rightsquigarrow$ $\mathrm{Ext}^i(\mathrm{Hom}_R(M,N))$ is well-defined (using projective resolutions of M) up to unique isomorphism!

<u>Define</u>

$$\mathrm{Ext}^i_R(M,N) = \varinjlim_{\substack{P_\bullet \to M \\ \text{proj. resol}^n \text{ of } M}} H^i(\mathrm{Hom}_R(P_\bullet, N))$$

(2) <u>Define</u> $\mathrm{Ext}^i_R(M,N)$

using injective resolutions of N:

$N \to I_\bullet$: $0 \to N \to I_0 \to I_1 \to \cdots \to$ exact

inj. resolution

injective left R-modules

$$\mathrm{Ext}^i_R(M,N) \underset{\text{def}}{=} H^i\left(\mathrm{Hom}^\bullet_R(M, I_\bullet)\right)$$

unique up to unique isom,

$$= \varinjlim_{\substack{N \to I_\bullet \\ \text{inj. resol}^n}} H^i(\mathrm{Hom}_R(M, I^\bullet))$$

(3) Use both: a proj. resol^n $P_\bullet \to M \to 0$

and an inj. resol $0 \to N \to I^\bullet$

$$\mathrm{Ext}^i_R(M,N) = H^i\left(\begin{array}{l}\text{the (simple) co-chain cpx} \\ \text{associated to the double} \\ \text{co-cochain complex} \\ \qquad \mathrm{Hom}_R(P_\bullet, I^\bullet)\end{array}\right)$$

<u>and</u> show that the third def^n coincides

with the first two !