Integral extensions, flatness

Associated primes & primary decompositions

Flatness: exactness w.r.t $\otimes$

its importance was realized later"
- by Serre

in the context of $\mathcal{O} \to \hat{\mathcal{O}}$

where $\mathcal{O}$ = a Noetherian commutative local ring

$\mathfrak{m}$ ← max. ideal

$\mathcal{O} \to \hat{\mathcal{O}}$ = the $\mathfrak{m}$-adic completion of $\mathfrak{m}$

$$= \varprojlim_{n \in \mathbb{N}} \mathcal{O}/\mathfrak{m}^{n+1}$$

$$= \varprojlim \left(\mathcal{O}/\mathfrak{m} \twoheadleftarrow \mathcal{O}/\mathfrak{m}^2 \twoheadleftarrow \mathcal{O}/\mathfrak{m}^3 \leftarrow \cdots \right)$$

= a Noetherian local ring, with maximal ideal $\mathfrak{m} \cdot \hat{\mathcal{O}}$

Fact $\mathcal{O} \to \hat{\mathcal{O}}$ is flat

Example $\mathbb{C}[x]$ $(x) = \mathbb{C}[x] \ni$ a maximal ideal

$\rightsquigarrow$ $\mathbb{C}[x]_{(x)} = \left\{ \begin{matrix} \text{the ring consisting of all rational} \\ \text{functions on } \mathbb{C} \text{ regular at } 0 \end{matrix} \right\}$

$\cap$

$\mathbb{C}\langle\langle x \rangle\rangle = \left\{ \begin{matrix} \text{all power series with } > 0 \\ \text{radii of convergence} \end{matrix} \right\}$

$= \left\{ \begin{matrix} \text{all germs of holomorphic functions} \\ \text{in open nbd's of } 0 \in \mathbb{C} \end{matrix} \right\}$

$\cap$

$\mathbb{C}[[x]]$

where $\mathbb{C}[\![x]\!] = \left\{ \sum_{n\in\mathbb{N}} a_n x^n \;\middle|\; a_n \in \mathbb{C} \;\; \forall n \in \mathbb{Z} \right\}$

$\parallel$

the completion of $\mathbb{C}[x]_{(0)}$ (w.r.t. their max. ideals)

$\parallel$

the completion of $\mathbb{C}\langle\langle x \rangle\rangle$

$$\begin{array}{l} \mathbb{C}[x]_{(0)} \to \mathbb{C}[\![x]\!] \\ \mathbb{C}\langle\langle x\rangle\rangle \to \mathbb{C}[\![x]\!] \end{array} \quad \text{are } \underline{\text{flat}}!$$

Why flatness is important.

$\underline{*}$: Many useful properties are preserved under flat extensions.

$M/\mathcal{O}$ has property $\mathbb{P}$ $\xrightarrow{\text{in many situations}}$ $M \otimes_{\mathcal{O}} \hat{\mathcal{O}}$

module over $\mathcal{O}$, algebra over $\mathcal{O}$, etc.

holds under certain situations/hypotheses (usually more restrictive)

"base change to $\hat{\mathcal{O}}$"

also has property $\mathbb{P}$

— $\mathcal{O} \to \hat{\mathcal{O}}$ faithfully flat

Noetherian local

(*) $0 \to M' \to M \to M'' \to 0$ complex of $\mathcal{O}$-modules

$\mathcal{O} \to \hat{\mathcal{O}}$ flat : (*) exact $\Rightarrow$ (*)$'$ is exact

(*)$'$ $0 \to M' \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to M \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to M'' \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to 0$

$\mathcal{O} \to \hat{\mathcal{O}}$ f.flat : flat + (*)$'$ is exact $\Rightarrow$ (*) is exact

$\underline{\text{Ex}}$

1) (complexes of $\mathcal{O}$-modules) exactness / acyclic (before and/or after base change)

2) smoothness (Jacobian of a finite number of polynomial equations over $\mathcal{O}$)

3) flatness

4) modules being of finite type
finite presentation

Grothendieck :

Let R be a commutative ring.
Consider the family of faithfully flat ring homomorphism

$$\{ R \xrightarrow{f} S \}.$$

There is a generalization of the idea/notion of topology (known as "Grothendieck topology") such these these families correspond to the usual notion of

$$\{ \text{open coverings of } \operatorname{Spec}(R) \}$$

"flat topology"

"Flatness spelled out"

A: commutative alg. M: A-module.

Suppose $a_1, \dots, a_r \in A$
$x_1, \dots, x_r \in M$
and $a_1 x_1 + \cdots + a_r x_r = 0$ $\Rightarrow$ $a_1, \dots, a_r$ gives a "syzygy" of $(x_1, \dots, x_r)$

Suppose moreover that M is flat over A.

$$0 \to N \xrightarrow{\gamma} A^{\oplus r} \xrightarrow{\beta} A$$

β defined by $(a_1, \dots, a_r)$

$$(b_1, \dots, b_r) \longmapsto a_1 b_1 + \dots + a_r b_r$$

$\Rightarrow$ as M is flat $/A$

$$0 \to N \otimes_A M \xrightarrow{\tilde{\gamma}} M^{\oplus r} \xrightarrow{\tilde{\beta}} M \quad \text{is exact}$$

$$(m_1, \dots, m_r) \longmapsto a_1 m_1 + \dots + a_r m_r$$

$\exists$ elements $n_1, \dots, n_k \in N$

s.t. $(x_1, \dots, x_r) = \tilde{\gamma}\left(\sum_{i=1}^{b} \underset{\in N}{n_i} \otimes \underset{\in M}{y_i}\right)$

Write $n_i = (b_{i1}, \dots, b_{ir}) \in N$

$$\sum_{j=1}^{r} b_{ij} \cdot a_j = 0 \quad \forall i = 1, \dots, b$$

$$\tilde{\gamma}\left(\sum_{i=1}^{b} n_i \otimes y_i\right) = \left(\sum_i y_i b_{i1}, \sum_i y_i b_{i2}, \dots, \sum_i y_i b_{ir}\right)$$

$$= (m_1, \dots \dots, m_r)$$

i.e $\begin{cases} m_j = \sum_{i=1}^{b} y_i b_{ij} \\ \sum_{j=1}^{r} b_{ij} a_j = 0 \end{cases} \xRightarrow[\text{trivially.}]{} \sum_{j=1}^{r} a_j m_j = 0$

Exercise The converse hold

i.e If this property holds then M is flat over A.

Nakayama's Lemma

$(A, \mathfrak{m})$ commutative local ring

If M is a finitely generated A-module

($\Leftrightarrow$ M is an A-module of finite type)

and $M \otimes_A (A/\mathfrak{m}) = 0$

$(\Longleftrightarrow M = \mathfrak{m} \cdot M)$

then: $M = 0$

Pf: Let $x_1, \dots, x_r$ be a set of A-generators of M

($\Rightarrow$ every element of M is an A-linear combination of $x_1, \dots, x_r$ with coeff. in $\mathfrak{m}$)

$\Rightarrow \forall_{i=1,\dots,r}$

$\exists\, a_{ij} \in \mathfrak{m}$

s.t. $x_i = \sum_{j=1}^{r} a_{ij} \, x_j$

Let $B = (a_{ij})_{1 \le i, j \le r}$

$$\underbrace{(I_n - B)}_{\substack{\parallel \\ C \\ \cap \\ M_n(A)}} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = 0.$$

Let D = the cofactor matrix of C

$$\rightsquigarrow \quad D \cdot C = \underset{\substack{\text{|||} \\ 1 \ (\mathrm{mod}\ \mathfrak{m})}}{\overset{\substack{A \\ \Downarrow}}{\det(C)}} \cdot I_n$$

$\rightsquigarrow \det(C) \in A^\times$

$$0 = D \cdot C \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = \underbrace{\det(C)}_{\in A^\times} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}$$

$\Longrightarrow$ $x_1 = \cdots = x_r = 0$ i.e. $M = 0$. q.e.d.

What we actually proved:

Suppose M is a finite A-module (i.e. M is finitely generated as an A-module)

$I \subseteq A$ an ideal

If $I \cdot M = M$, then $\exists$ an element $u \in A$ s.t. $u \equiv 1 \pmod{I}$ and $u \cdot M = 0$ (i.e. $\underbrace{\mathrm{Ann}_A(M)}_{\overset{\text{def}}{=} \{x \in I \mid x M = 0\}} \ni u$)

Associated primes, primary decomposition

"old fashioned" statement.

A: a commutative Noetherian ring.

Every ideal I of A admits a primary decomposition

i.e. $I = Q_1 \cap \cdots \cap Q_m$

where each Q_i is a primary ideal

Recall an ideal Q of A is primary for a prime ideal P of A if

1) $x \cdot y \in Q$, and $x \notin Q$, then $y \notin P$

   $\forall x, y \in A$

2) $\forall$ element $x \in Q$ $\exists n \in \mathbb{N}$ s.t. $x^n \in P$

Ex p : a prime number.

$p^i \mathbb{Z}$ is p-primary $\forall i \in \mathbb{N}_{\geq 1}$

$\mathbb{Z}[\sqrt{-5}]$ not a PID