Some examples / further questions related to tensor products.

Tensor products from a free module M algebras of finite rk over a commutative ring R.

$$R[x_1, \dots, x_n] = S_R^{\cdot}(\underbrace{Rx_1 \oplus \cdots \oplus Rx_n}_{\text{free } R\text{-module with free generators } x_1,\dots,x_n})$$

Say $R = \mathbb{Z}$

Have an incarnation of $\mathbb{Z}[x_1, \dots, x_n]$:

$$\mathbb{Z}[\partial_1, \dots, \partial_n] \subseteq \mathbb{C}\left[\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right]$$

$$\mathbb{Z}[x_1,\dots,x_n] \cong \mathbb{Z}[\partial_1, \dots, \partial_n] = \left\{ \sum_{\substack{I \subseteq \mathbb{N}^n \\ \text{finite}}} a_I \, \partial^I \,\middle|\, a_I \in \mathbb{Z} \ \forall I \right\}$$

$$\mathbb{Z}[\partial_1, \dots, \partial_n] \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[u_1, \dots, u_n], \mathbb{Z}[u_1, \dots, u_n])$$

$$P(\partial_1, \dots, \partial_n) \longmapsto (f(\underline{u}) \mapsto Pf)$$

Are there other such differential operators with rational constant coeff which send $\mathbb{Z}[\underline{u}]$ to $\mathbb{Z}[\underline{u}]$

Example $\dfrac{1}{k_1! \cdots k_n!} \partial_1^{k_1} \cdots \partial_n^{k_n}$

$$S^{\bullet}_{\mathbb{Z}}(\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n) = \mathbb{Z}[x_1, \dots, x_n]$$

$\nearrow$ ← quotient by the ideal gen. by $(x_i \otimes x_j - x_j \otimes x_i)$, $1 \le i, j \le n$

$$T^{\bullet}_{\mathbb{Z}}(\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n)$$

$$\cup\!\!\mid$$

$$\mathcal{S}\left(T^{\bullet}_{\mathbb{Z}}(\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n)\right) \overset{\text{defn}}{=} \bigoplus_{m \in \mathbb{N}} \left(T^{m}_{\mathbb{Z}}(\mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n)\right)^{S_m}, \text{ a subring.}$$

non-commutative if $n \ge 2$

<u>Question</u>: Is there a natural ring structure on $\mathcal{S}(T^{\bullet}_{\mathbb{Z}} M)$, $M = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$.

Example of how $\otimes$-product is used.

k: a field

$$\mu : \underbrace{k[t_1, \dots, t_m]}_{P} \otimes_k \underbrace{k[t_1, \dots, t_m]}_{P} \longrightarrow \underbrace{k[t_1, \dots, t_m]}_{P}$$

μ: product

$k[t_1, \dots, t_m]$: graded by $\mathbb{N}$

$k[t_1, \dots, t_m]$: graded vector space, graded by degree

$$= \bigoplus_{w \in \mathbb{N}} \left(\text{homog. poly. of deg } w \text{ over } k\right)$$

dualize (over k) $\qquad P = \bigoplus_{w \in \mathbb{N}} P_w$

Let $\quad P^* =$ the graded dual of P

$$\overset{\text{def}}{=} \bigoplus_{w \in \mathbb{N}} \underbrace{\mathrm{Hom}_k(P_w, k)}_{} = Q^w$$

$$\rightsquigarrow \quad \underset{\Delta}{\overset{\parallel}{\mu^*}} : \underset{Q}{\overset{\parallel}{P^*}} \longrightarrow \overset{Q \otimes_k Q}{\overset{\parallel}{P^* \otimes_k P^*}}$$

Both factors P^* are graded, which gives/induces a grading on $P^* \otimes_k P^*$

$P_1 \ni t_1, \dots, t_m$. a basis of P_1

$Q^1 \ni \lambda_1, \dots, \lambda_m$: dual basis of $t_1, \dots, t_m$

$$\begin{array}{cccc} \Delta : & Q & \longrightarrow & Q \otimes_k Q \\ \parallel & \Downarrow & & \\ \mu^* & \lambda_1, \dots, \lambda_n & & \\ Q^1 \Rightarrow & \lambda_i & \xmapsto{\Delta} & ? \quad \underline{\overset{\deg = 1}{\lambda_i} \otimes \overset{\deg = 1}{\lambda_i}} \end{array}$$

$$\mu : \quad P \otimes_k P = \bigoplus_{w \in \mathbb{N}} \underbrace{\bigoplus_{\substack{i,j \in \mathbb{N} \\ i+j=w}} P_i \otimes_k P_j}_{\substack{\parallel \\ (P \otimes_k P)_w}} \longrightarrow \bigoplus_{w \in \mathbb{N}} P_w$$

$$\begin{array}{ccc} P_0 \otimes P_1 \oplus P_1 \otimes P_0 & & \\ \parallel & & \\ (P \otimes_k P)_1 & \longrightarrow & P_1 \end{array}$$

its dual is : $\quad Q^0 \otimes Q^1 \oplus Q^1 \otimes Q^0 \longleftarrow Q^1$

$$P_0 = k \qquad Q_0 \overset{\text{"}}{=} k \qquad \underline{(k\otimes Q^1)\oplus(Q^1\otimes k)} \xleftarrow{\ \Delta\ } Q^1$$

≥ 1

$$P = S^{\bullet}_{k}(kx_1\oplus\cdots\oplus kx_n)^{*} \leftarrow \text{graded dual}$$

$$= \bigoplus_{w\in\mathbb{N}} \mathrm{Hom}_k\left(S^w_k(kx_1\oplus\cdots\oplus kx_n), k\right)$$

$$\mu: P\otimes_k P \longrightarrow P$$

<u>Exer</u> i) $1\otimes\lambda_i + \lambda_i\otimes 1 \longleftarrow \lambda_i$

iii)

$$\begin{tikzcd} & Q \arrow[r, "\Delta"] \arrow[dl, "\Delta"'] & Q\otimes_k Q \arrow[dd, "\Delta\otimes 1"] \\ Q\otimes Q \arrow[d, "1\otimes\Delta"'] & & \\ Q\otimes_k(Q\otimes_k Q) \arrow[rr, "\cong"] & & (Q\otimes_k Q)\otimes_k Q \end{tikzcd}$$

Is the above diagram commutative?

ii) Find a natural arrow

$\varepsilon: Q \longrightarrow k$

recall $Q = \bigoplus_{w\in\mathbb{N}} \underbrace{\mathrm{Hom}_k(P_w, k)}_{= Q^w}$

such that the diagram $\longrightarrow$ commutes!

$$\begin{tikzcd} Q \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "1_Q"] & Q\otimes_k Q \arrow[d, "\varepsilon\otimes 1"] \\ Q\otimes_k Q \arrow[r, "1\otimes\varepsilon"'] & Q\otimes_k k = k\otimes_k Q = Q \end{tikzcd}$$

$k[t_1, \dots, t_m] \longrightarrow k \qquad \varepsilon \in Hom_k(Q, k)$

<u>Hint</u> (iii) is dual to associativity of μ.

(ii) is dual to the existence of unity.

Dualize the arrows of the structure map of an algebra, you get a new algebraic structure
a product

<u>"co-algebra"</u>

<u>Example of</u> $\mathbb{Z}[x_1, \dots, x_n]$:

1) has a well-known structure as an algebra (over $\mathbb{Z}$)

2) has a <u>compatible</u> structure as a co-algebra

<u>so</u> that the product map is compatible with coproduct map

$$\boxed{\mathcal{S}\left(T_{\mathbb{Z}}(x_1 \mathbb{Z}_1 \oplus \cdots \oplus x_n \mathbb{Z}_n)\right)}$$

has a natural ring structure

$i, j \in \mathbb{N}$

$$\bigotimes_R^i M \supseteq \underbrace{\left(\bigotimes_R^i M\right)^{S_i}}_{\text{symmetric tensors in } \bigotimes_R^i M} \times \underbrace{\left(\bigotimes_R^j M\right)^{S_j}}_{\text{symmetric tensors in } \bigotimes_R^j M} \subseteq \bigotimes_R^j M$$

$$\downarrow$$

$$\left(\bigotimes_R^{i+j} M\right)^{S_{i+j}}$$

$$\left(\bigotimes_R^i M\right) \otimes \left(\bigotimes_R^j M\right) \longrightarrow \bigotimes_R^{i+j} M$$

How to produce a symmetric $(i+j)$-tensor from a symm i-th tensor and a symm. j-th tensor?

<u>Hint</u>: <u>symmetrize</u>!