"Basic commutative algebra".

R : commutative ring

$\rightsquigarrow$ $\mathrm{Spec}(R)$ + Zariski topology $\leftarrow$ $U_f = \mathrm{Spec}(R[\frac{1}{f}])$

has structure sheaf $\mathcal{O}$ $\overset{\text{non-nilp}}{\nearrow}$

$\mathcal{O} = \mathcal{O}_{\mathrm{Spec}(R)}$

(Otherwise $R[\frac{1}{f}]$ is the 0 ring, and $\mathrm{Spec}(R[\frac{1}{f}]) = \emptyset$ by def")

$= \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p} \}$

$R \xrightarrow{\sim} \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$

$R[\frac{1}{f}] \xrightarrow{\sim} \Gamma(U_f, \mathcal{O}_{\mathrm{Spec}\, R})$

a sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules

Moreover, every R-module defines a sheaf $\widetilde{M}$ on $\mathrm{Spec}(R)$ such that

$$\Gamma(U_f, \widetilde{M}) = M[\tfrac{1}{f}].$$

In particular, $M \xrightarrow{\sim} \Gamma(\mathrm{Spec}(R), \widetilde{M})$

Example: $R = \mathbb{C}[x_1, x_2]$ polynomial ring.

$\mathrm{Spec}(R) \ni \mathbb{C}^2$

$\mathfrak{m}_{(a,b)} \longleftarrow (a,b)$

$\mathfrak{m}_{(a,b)} = (x-a, y-b) = \ker(ev_{a,b}; f \mapsto f(a,b))$

Let $U = R \setminus \{ P_{(0,0)} \}$ is open subset of $\mathrm{Spec}(R)$.

Question: What is $\Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)})$?

max. ideals $\rightsquigarrow$ $\{\mathfrak{m}_{a,b}\}$ is closed $\forall (a,b) \in R$

Let $P_{(0,0)} = \mathfrak{m}_{(0,0)}$

$x-1 \notin \mathfrak{m}_{(0,0)} \rightsquigarrow \frac{1}{x-1} \in \mathbb{C}[x,y]_{\mathfrak{m}_{(0,0)}}$

does NOT define a global section on U!

$\frac{1}{x-1}$ has a "simple pole" along $\{1\} \times \mathbb{C}$

What are $\dfrac{f(x,y)}{g(x,y)}$ which are regular / holomorphic can be extend to outside of $\{(0,0)\}$

rat'l function

$f, g \in \mathbb{C}[x,y]$, $g \neq 0$ $\quad$ $g(x,y)$ does not vanish identically on $\mathbb{C}^2$

$\rightsquigarrow$ Let $V = \{ (a,b) \in \mathbb{C}^2 \mid g(a,b) \neq 0 \}$ $\quad$ $\neq \phi$ open

$\rightsquigarrow$ $\dfrac{f(x,y)}{g(x,y)}$ defines a holomorphic function on V

Let $\tilde{V}$ = the largest open subset containing V s.t. $\exists$ a holomorphic function $\hat{h}$ on $\tilde{V}$ s.t. $\tilde{h}|_V = \dfrac{f}{g}$

$\tilde{V} = V_{\frac{f}{g}}$

Question Which $\dfrac{f}{g} \in \mathbb{C}(x,y)$ has the property that $V_{\frac{f}{g}} \supseteq \mathbb{C}^2 \setminus \{(0,0)\}$.

Ans / Exer.

$$\Gamma\left(\operatorname{Spec}(\mathbb{C}[x,y]) \setminus \{\mathfrak{m}_{(0,0)}\},\ \mathcal{O}_{\operatorname{Spec} \mathbb{C}[x,y]} \right)$$
$$\uparrow \wr$$
$$\mathbb{C}[x,y]$$

Complex holomorphic version Hartogs theorem

Special case: every holomorphic function on $\{(z,w)\in\mathbb{C}^2 \mid |z|<1, |w|<1\} \setminus \{(0,0)\}$

extends to a holomorphic function on $\{(z,w)\in\mathbb{C}^2 \mid |z|,|w|<1\}$

* The polar locus of a rational function on $\mathbb{C}^2$ is a divisor!

ex.

$\mathbb{C}[x,y]_{\mathfrak{m}_{(0,0)}}$ is a local ring, i.e. has just one maximal ideal.

$$\mathbb{C}[x,y]_{\mathfrak{m}_{(0,0)}} = \left(\mathbb{C}[x,y]\setminus \mathfrak{m}_{(0,0)}\right)^{-1}\cdot \mathbb{C}[x,y]$$

$$= \varinjlim_{\substack{f\in\mathbb{C}[x,y]\\ f(0,0)\neq 0}} \mathbb{C}[x,y,\tfrac{1}{f}]$$

$$= \varinjlim_{\substack{f\in\mathbb{C}[x,y]\\ f(0,0)\neq 0}} \Gamma(U_f, \operatorname{Spec}\mathcal{O}_{\mathbb{C}[x,y]})$$

such open subsets U_f with $f(0,0)\neq 0$ form a basis of open neighborhoods of $\mathfrak{m}_{(0,0)}$

= the germ of all rational functions which are regular at $\mathfrak{m}_{(0,0)} \in \operatorname{Spec}\mathbb{C}[x,y]$.

$\mathbb{R}^2 \ni (0,0)$ $\varinjlim \mathbb{C}^\infty(U, \mathbb{R})$ = germ of C^∞-functions at $(0,0) \in \mathbb{R}^2$

open neighborhoods of $(0,0)$

$\mathbb{C} \ni 0.$ $\varinjlim_U$ {holomorphic function on an open nbd of $0 \in \mathbb{C}$}

= germ of holomorphic functions at $0 \in \mathbb{C}$

$= \left\{ \sum_{n\in\mathbb{N}} a_n z^n \;\middle|\; \begin{array}{l} a_n \in \mathbb{C} \ \forall n \in \mathbb{N} \\ \text{radius of convergence} > 0 \end{array} \right\}$

$R_{\mathfrak{z}}$ = "the germ of all rational function which are regular in an open nbd of $\mathfrak{z} \in \operatorname{Spec}(R)$"

$\mathfrak{z} \in \operatorname{Spec} R$

$\{U : U \text{ open nbd of } 0 \text{ in } \mathbb{C}\} =: \mathcal{U}$

$\varinjlim_{U\in\mathcal{U}} \Gamma(U, \mathcal{O}_{\mathbb{C}}^{hol})$ ← holomorphic functions on U

$= \coprod_{U\in\mathcal{U}} \Gamma(U, \mathcal{O}_{\mathbb{C}}^{hol}) \Big/ \Big($ the equiv relation

$f_1 \sim f_2$

$\Gamma(U_1, \mathcal{O}_{\mathbb{C}}^{hol}) \quad \Gamma(U_2, \mathcal{O}_{\mathbb{C}}^{hol})$

if $\exists\, U_3 \subseteq U_1 \cap U_2$

$\mathcal{U}$ open nbd of 0 in $\mathbb{C}$

s.t. $f_1|_{U_3} = f_2|_{U_3}$ $\Big)$

Example

Lemma / Exercise M : R-module. $(R \setminus \mathfrak{p})^{-1} M$

$M \ni m$

If the image of m in $M_{\mathfrak{p}}$ (under the natural map $M \longrightarrow M_{\mathfrak{p}}$) for every $\mathfrak{p} \in$ ~~Spec R~~ Max R, then $m = 0$ (in M).

$M \ni m \rightsquigarrow$ an element $\widetilde{m}$ of $\Gamma(\mathrm{Spec}(R), \widetilde{M})$

Condition: $\forall$ point $\mathfrak{p} \in \mathrm{Spec}\, R$, the germ of $\widetilde{m}$ at $\mathfrak{p}$ is 0.

$\rightsquigarrow \widetilde{m}$ vanishes in an open subset $U(\mathfrak{p})$ $\forall \mathfrak{p} \in \mathrm{Spec}(R)$

$\Longrightarrow \widetilde{m} = 0$!

Pf: Consider the subset $S = \{ s \in R \mid s \cdot m = 0 \}$

$R \supseteq \; = \mathrm{Ann}_R(m) = \text{an ideal of } R$.

Assumption on m

$\Longleftrightarrow$ I is not contained in any maximal ideal of R

$\Longrightarrow I = R$. q.e.d.