Contine on induced repr.

More General context:

R, S : rings

$h: R \to S$ a ring homomorphism

M: left R-module

$\rightsquigarrow$ $\underset{(h,R)}{S \otimes M}$: is a left S-module

a (S, R)-bimodule via $h: R \to S$

Special case $H \leq G$, subgroup H ← group G; k: a field

$\rightsquigarrow$ $k[H] \hookrightarrow k[G]$

(V, ρ): k-linear repr. of H

$\rightsquigarrow$ $\mathrm{Ind}_H^G(V, \rho) :=$ the left $k[G]$-module $k[G] \underset{k[H]}{\otimes} V$.

Assume $\dim_k(V) < \infty$

$$\boxed{\chi_{\mathrm{Ind}_H^G(V,\rho)}(x) = \sum_{\substack{x \cdot y_i H = y_i H}} \overbrace{\chi_\rho(y_i^{-1} x y_i)}^{\text{depends only on } y_i H}}$$

$x \cdot y_i H = y_i H \iff y_i^{-1} x \cdot y_i \in H$ (depends only on $y_i H$)

Choose representative $y_1, \dots, y_r$ of G/H

i.e. $G = \coprod_{i=1}^{r} y_i H$

__Remark__

If $\#H \cdot 1_k \in k^\times$, then

$$\boxed{\mathrm{Ind}_H^G(\chi_\rho) = \frac{1}{\#H} \sum_{\{y \in G \mid y^{-1}xy \in H\}} \chi_\rho(y^{-1} x y)}$$

__Generally:__ (last time)

$\mathrm{Ind}_H^G(\mathbb{1}_H) = k[G/H]$ with left translation action by G on G/H

if $H \neq G$ $\Rightarrow$ $\not\cup$ $k \cdot \left(\sum_{y \in G/H} [y]\right)$ $\Rightarrow$ $\mathrm{Ind}_H^G(\mathbb{1}_H)$ is reducible

Sometimes, $\mathrm{Ind}_H^G(\mathbb{1}_H)$ is irreducible

__Ex.__ $(\mathbb{Z}/3\mathbb{Z}) \triangleleft D_6$

$(\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\chi} \mathbb{C}^\times$

nontrivial (2 such character)

$\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{D_6}(\chi)$ is __irreducible__ !

"Mackey's criterion" G: a finite group.

Suppose $N \trianglelefteq G$, (V, ρ): irreducible $\mathbb{C}$-repr of V

$\mathrm{Ind}_N^G(V,\rho)$ is irreducible (or k = alg. closed field, $\#G \cdot 1_k \in k^\times$)

iff $y_1, y_2 \in G$, the representation

$$\tau_i : N \xrightarrow{\mathrm{Ad}(y_i^{-1})} N \xrightarrow{\rho} GL(V) \qquad i = 1, 2$$

are isomorphic if and only if $y_1 \in y_2 N$.

($\Longrightarrow$ always true)

<u>Exercise</u> for the formula of induced character.

<u>i.e.</u> use the previous formula for $\mathrm{Ind}_N^G(\rho)$ to compute/express $\mathrm{Ind}_N^G(\chi_\rho)$ and $\left(\mathrm{Ind}_N^G(\chi_\rho) \middle| \mathrm{Ind}_N^G(\chi_\rho)\right)$.

$= 1$ iff $\mathrm{Ind}_N^G(V,\rho)$ is irreducible

<u>Illustration</u>: $D_6 \trianglerighteq (\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\chi} \mathbb{C}^\times$ (irrd)

Let $\tau \in D_6$ be a reflection

The conjugate of χ by τ is $\overline{\chi} \neq \chi$.

Mackey $\Longrightarrow$ $\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{D_6}(\chi)$ is an irred. 2-dim$^{\underline{l}}$ repr. of D_6.

<u>Exer</u> p: prime nber

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\}$$

$\#H = p^3$

finite Heisenberg group $\leq GL_3(\mathbb{F}_p)$

has many subgroups.

e.g.

$$N = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, c \in \mathbb{F}_p \right\}$$

abelian

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; c \in \mathbb{F}_p \right\}$$

$Z(H) \subsetneqq N$

$N/Z(H)$ has p 1-dim$^{\underline{l}}$ characters $\mathbb{1}_{N/Z(H)} = \chi_1, \dots, \chi_p$

(<u>Remark</u>: Every finite commut. group G has $\#G$ 1-dim$^{\underline{l}}$ characters $G \to \mathbb{C}^\times$)

Inflate $\chi_1, \dots, \chi_p$ to N (i.e. precompose with $N \twoheadrightarrow N/Z(H)$),

get p induced repr.

$\mathrm{Ind}_N^H(\chi_i)$, $i = 1, \dots, p$.

Determine: whether $\mathrm{Ind}_N^H(\chi_i)$ is irred for $i = 2, \dots, p$

Key (Fix $\overset{2}{\underset{1}{\#}}$) $h \in H$
$n \mapsto \chi_i(h^{-1} n h)$
$\overset{\cap}{N}$

non-trivial 1-dim.^{l} character of N.

When is $n \mapsto \chi_i(h_1^{-1} n h_1)$ and $n \mapsto \chi_i(h_2^{-1} n h_2)$ equal?

Jordan's Theorem $\forall n \in \mathbb{N}_{>0}$, $\exists$ a const $C_n > 0$ s.t. $\forall$ finite subgroup $G \leqslant GL_n(\mathbb{C})$, or, any field of char. 0, $\exists$ an abelian normal subgroup $H \trianglelefteq G$ with $[G : H] \leqslant C_n$.

Note: May choose
$$C_n = (\sqrt{8n}+1)^{2n^2} - (\sqrt{8n}-1)^{2n^2}$$

Uses $GL_n(\mathbb{C}) \supseteq U_n$ = standard unitary group

Sketch of proof

$(A,B) = AB A^{-1} B^{-1}$

(*) **Lemma 1**: Let $A, B \in U_n$. If $(A,(A,B)) = I_n$ and $\|I_n - B\| \leq 2$, then $(A,B) = 1$

standard norm on $U_n \subseteq M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ i.e. $\|C\| = (\mathrm{Tr}\, C \cdot {}^t\bar{C})^{1/2}$, $C \in M_n(\mathbb{C})$

Lemma 2 If $A, B \in U_n$, then

"easy"

$$\|I_n - (A,B)\| \leq \sqrt{2}\, \|I_n - A\| \cdot \|I_n - B\|$$

Lemma 1 + Lemma 2

Lemma 3 If $A, B \in$ a finite subgroup G of U_n and $\|I_n - A\| < \frac{1}{\sqrt{2}}$, $\|I_n - B\| < 2$

then $AB = BA$.

Lemmas 1+2+3 $\Longrightarrow$ Jordan's theorem.

Idea Use: H = the subgroup of G generated by all elements $A \in G$ s.t. $\|I_n - A\| < \frac{1}{\sqrt{2}}$

normal, and abelian

Find an easy bound of $[G : H]$