Discussed: tensor products
symmetric product,
exterior product.

— modules

More about modules. ACC DCC
$\nearrow$ $\nearrow$
chain conditions

Def: Let R be a ring, M be a (left) R-module.

1) M satisfies the ACC, or, M is a Noetherian left R-module, if one of the following equivalent conditions hold

- every submodule $N \subseteq M$ is finitely generated
- every ascending chain of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots\cdots$ stops
  i.e. $\exists\, n_0 \in \mathbb{N}$ s.t. $N_{n_0+i} = N_{i_0}$ $\forall\, i \geq 0$
- Every family of submodules of M has (at least) a maximal element.

Similarly

2) M satisfies the DCC (terminology M is an Artinian left R-module) if every descending chain of R-submodules stops., or equivalently, every family of submodules of M has a minimal element.

Def: A ring R is left $\begin{cases}\text{Noetherian}\\ \text{Artinian}\end{cases}$ if the free left R-module R is $\begin{cases}\text{Noetherian}\\ \text{Artinian}\end{cases}$

Similary for right.

Remark:
- Both are "finiteness conditions"
- Being Artinian is very stringent

  e.g. Have good structure theorem for Artinian ring (either commutative or non-commutative)

$$\underbrace{\underset{\nearrow\,\text{Artinian}}{R} \Big/ \underset{\text{radical}}{\text{Jacobson}}(R)}_{\text{semi-simple Artinian}} = \text{a finite product of rings of the form } M_n\left(\text{division ring}\right)$$

- "Often": Artinian $\Longrightarrow$ Noetherian

Example: $R = \mathbb{C}[x, y]$ — Noetherian — Not Artinian

An R-module M is Artinian

$$\Longleftrightarrow \quad \dim_{\mathbb{C}}(M) < \infty$$

Basic Properties

$$0 \to M' \to M \to M'' \to 0$$ short exact sequence of R-modules

a) if M' and M'' are both $\begin{cases}\text{Noetherian}\\ \text{Artinian}\end{cases}$, then so is M

b) M Artinian $\Longrightarrow$ both M' and M'' are

c) M Noetherian

$\overset{?}{\Longrightarrow}$ M' Noetherian <u>Yes</u>.

$\overset{?}{\Longrightarrow}$ M'' Noetherian <u>Yes</u>

<u>Question</u> $R_1 \leqslant R_2$ rings

R_2 Noetherian.

$\overset{?}{\Longrightarrow}$ R_1 is Noetherian <u>NO</u>

$$R = \mathbb{C}[x_n \mid n \in \mathbb{N}] \subsetneq \mathbb{C}(x_n \mid n \in \mathbb{N}) = \operatorname{frac}\left(\mathbb{C}[x_n \mid n \in \mathbb{N}]\right)$$

<u>NOT</u> Noetherian:

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots \subsetneq (x_1, \cdots, x_n) \subsetneq$$

<u>Proposition</u>: R: commutative. Noetherian

$\Longrightarrow$ $R[x]$ = poly. ring in one variable over R

<u>Then</u>: $R[x]$ is <u>Noetherian</u>

("Exer.")

<u>hint</u>: Given an increasing chain (I_i) of ideals in $R[x]$, look at the corresponding ideal of leading coeff. in I_i.

$\Longrightarrow$ $R[x_1, \cdots, x_n]$ is Noetherian

$\Rightarrow$ Any finitely generated R-algebra is Noetherian.

($\because$ a quotient ring of a $\overset{\text{left}}{\text{right}}$ Noetherian ring is $\overset{\text{left}}{\text{right}}$ Noetherian)

Open ended question:

What about R non-comm.?

"What do you mean by "polynomial ring over R""

Functors: R: a ring.

$$\mathrm{Hom}: \underset{\overset{\shortparallel}{\ell\mathrm{Mod}_R}}{(\text{left } R\text{-modules})} \times \overset{\ell\mathrm{Mod}_R}{\overset{\shortparallel}{(\text{left } R\text{-modules})}} \to \overset{\mathrm{AbGrp}}{(\text{abel. groups})}$$

$$(M \quad , \quad N) \rightsquigarrow \mathrm{Hom}_R(M, N)$$

Fix M:

$$\mathrm{Hom}_R(M, -): \ell\mathrm{Mod}_R \longrightarrow \mathrm{AbGrp}$$

$$N \longrightarrow \mathrm{Hom}_R(M, N)$$

Is it exact?

$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to$ short exact.

? Is $0 \to \mathrm{Hom}_R(M, N') \overset{\text{inj}}{\underset{\alpha ?}{\to}} \mathrm{Hom}_R(M, N) \overset{\text{surj}}{\underset{\beta ?}{\to}} \mathrm{Hom}_R(M, N'') \cancel{\to} 0$

also short exact?

$\beta \circ \alpha = 0$ i.e. $\mathrm{Ker}(\beta) \supseteq \mathrm{Im}(\alpha)$

$\mathrm{Ker}(\beta) \overset{?}{=} \mathrm{Im}(\alpha)$

Ans: α is inj, $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$ but β may NOT be surjective!

Exer: Find a counter-example with $R = \mathbb{Z}$ and,
(easy)

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

p: a prime number

" $\mathrm{Hom}(M, \cdot)$ is left exact.

Similarly

contravariant

$$\mathrm{Hom}_R(\cdot, N) : \ell\mathrm{Mod}_R \longrightarrow \mathrm{AbGrp}.$$

$$M \rightsquigarrow \mathrm{Hom}_R(M.N)$$

$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ short exact

Is $0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{\alpha} \mathrm{Hom}_R(M, N) \xrightarrow{\beta} \mathrm{Hom}_R(M, N') \cancel{\to 0}$ exact?

$\mathrm{Hom}_R(\cdot, N)$ is left exact!

i.e. α is injective. and $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$

Exer. Find an example where β is NOT surj!

Def: a) A left R-module M is projective if $\mathrm{Hom}_R(M, -)$ is exact.

ex. free modules are exact

b) A left R-module N is <u>injective</u> if $\mathrm{Hom}_R(\cdot\ ,N)$ is exact.