Recall: We computed the character table of S_4 last time.

$$\text{the permutation repr. of } S_4 \text{ on } \mathbb{C}^4 \cong \mathbb{1} \oplus \underset{\text{3-dim}^{\text{l}}\text{ irred}}{U}$$

Found 4 irred repr.: $\mathbb{1}$, sgn, U, $U \otimes \text{sgn}$

+ an irred 2-dim$^{\text{l}}$ character χ_5

<u>Question:</u> Construct the linear repr. ρ_5 with character χ_5.
Study properties of ρ_5

<u>Example</u> S_4:

	1 (1)	6 $(1,2)$	8 (123)	3 $(12)(34)$	6 (1234)
$\mathbb{1} = \chi_1$	1	1	1	1	1
$\text{sgn} = \chi_2$	1	-1	1	1	-1
$\chi_U = \chi_3$	3	1	0	-1	-1
$\text{sgn}\cdot\chi_U = \chi_4$	3	-1	0	-1	1
χ_5	2	0	-1	2	0
eigenvalues of $\rho_5(z_j)$	1,1	1,-1	roots of $x^2+x+1=0$	1,1 $\uparrow$ i.e. $\rho((12)(34)) = I_2$	either $\cancel{\pm i}$ or $\boxed{\pm 1}$

$(1234)^2 = (13)(24)$

$\rho_5((13)(24))$ has eigenvalues 1, 1 $\Longleftarrow$ if $\rho_5((1234))$ has eigenvalues ± 1

not possible! $\rightarrow$ $\rho_5((13)(24))$ has eigenvalues $-1,-1$ $\Longleftarrow$ if $\rho_5((1234))$ has eigenvalues $\pm\sqrt{-1}$

Have seen: $\rho_3\big((12)(34)\big) = I_2$ $\mathcal{K}$

i.e. $\operatorname{Ker}(\rho_3) \supseteq$ the $\underbrace{\text{Klein-four}}_{C_4 \cup \{1_{S_4}\}}$ subgroup of S_4

$$S_4 \xrightarrow{\rho_5} GL(W)$$

W: 2-dim^l $\mathbb{C}$-vector space

$\operatorname{Ker}(\rho_5) \supseteq \mathcal{K}$

$$\rightsquigarrow \quad \underbrace{S_4/\mathcal{K}}_{\substack{\cong \\ S_3}} \xrightarrow{\overline{\rho_5}} GL(W) \quad \underline{\text{irred}}$$

order $= 6$

non-commutative (Schur's Lemma)

$$1 \to \mathcal{K} \to S_4 \to \underset{\substack{\cong \\ S_3}}{S_4/\mathcal{K}} \to 1$$

$S_4/\mathcal{K}$ operates by conjugation on $\mathcal{K} \setminus \{1\}$

$$S_4/\mathcal{K} \xrightarrow{\sim} \operatorname{Perm}(\mathcal{K} \setminus \{1\})$$

Have $S_3 \overset{j}{\hookrightarrow} S_4$ and $S_4 \cong \mathcal{K} \rtimes S_3$

$D_6 \cong S_3$ has 1 irred 2-dim^l repr.

Induced representations:

G : a finite group

$\vee\!\!/$

H : a subgroup of G

k : a field

τ: repr. of G

$\tau|_H$ = restriction of τ to H is a repr. of H

$H \hookrightarrow G \xrightarrow{\tau} GL_k(W)$

V: left $k[H]$-module, i.e a k-linear repr. of H

$\Leftrightarrow \rho : H \longrightarrow GL_k(V)$

Def: $\mathrm{Ind}_H^G(\rho, V) = k[G] \otimes_{k[H]} V$

a left $k[G]$-module

$k[H] \hookrightarrow k[G]$

a $(k[G], k[H])$-bimodule.

= the induced repr, from H to G, of (V, ρ)

Ex. $\{1\} \subseteq G$ $\qquad \mathrm{Ind}_{\{1\}}^G(k) = k[G] \otimes_{k[\{1\}]} k \xrightarrow{\simeq} k[G]$

k: trivial repr. of $\{1\}$; $\quad k[\{1\}] = k$

$\cong$ the left regular repr. of G, i.e regarding $k[G]$ as a left module over itself.

Ex. $S_3 \cong D_6$ has 2-dim$^{\text{l}}$ irred. repr.

$\vee\!\!/$

$\mathbb{Z}/3\mathbb{Z} \xrightarrow{\chi_i} \mathbb{C}^\times \qquad \chi_i(a \bmod 3) = \zeta^{ia}$, $\quad \zeta = e^{2\pi\sqrt{-1}/3}$

$i = 0, 1, 2$

$\underline{i=0}$: χ_0 = trivial character of $\mathbb{Z}/3\mathbb{Z}$

χ_1, χ_2 : conjugate pair of non-trivial irred. characters of $\mathbb{Z}/3\mathbb{Z}$

(a) $\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{D_6}(\chi_0) = ?$

<u>More generally</u>

How to think about

$\underbrace{H \leq G}_{\text{finite groups}}$ $\qquad \mathrm{Ind}_H^G(\mathbb{1}_H) \underset{\text{Exercise}}{\cong} \underbrace{k[G/H]}_{G \text{ operates on the left of } G/H}$

$\rightsquigarrow$ get a permutation repr. of G on $\underbrace{k[G/H]}_{\text{as the vector space}/k \text{ with basis } G/H}$

<u>Exer</u> $\boxed{\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{D_3}(\chi_i) \cong \text{the 2-dim}^{\ell} \text{ irred. repr of } D_3 \quad i=1 \text{ or } 2}$

<u>Question</u>: How to express $\chi(\mathrm{Ind}_G^H(\rho, V))$ in terms of χ_ρ

χ_ρ: a class function on H

$\chi(\mathrm{Ind}_G^H(\rho,V)) \overset{\text{def}}{=} \underbrace{\mathrm{Ind}_H^G(\chi_\rho)}_{\text{a class function on } G}$.

$$\mathrm{Ind}_H^G(\chi_\rho)(\underset{G}{x}) = \mathrm{Tr}\left(\text{left multiplication by } x \,\Big|\, \underbrace{k[G] \otimes_{k[H]} V_\rho}_{\text{vector space}/k}\right)$$

Let $\{y_i\}_{i=1}^m$ be a set of repr. of G/H

$k[G]$ = a free right $k[H]$-module with basis

$$\left(\coprod_{i=1}^{m} y_i \cdot H = G \quad \rightsquigarrow \quad \bigoplus_{i=1}^{m} y_i \, k[H] = k[G] \right)$$

x ~~operates~~ permutes on the free rk-one right $k[H]$-modules

$$\underbrace{y_i \, k[H]}_{\text{depends only on } y_i H} = k[y_i H]$$

$\rightsquigarrow$ The matrix repr. of x on $\mathrm{Ind}_H^G(V_p)$ w.r.t in terms of the above subspaces $k[y_i H]$, is in "block form",

"permutation block form"

i.e. $x \sim (m \times m)$-block form.

and only one non-zero entry in each horizontal/vertical direction.

m=3
$$\begin{bmatrix} 0 & \boxed{*} & 0 \\ 0 & 0 & \boxed{*} \\ \boxed{*} & 0 & 0 \end{bmatrix}$$
allowed

And consequently:

$\forall\ \underline{y_i H}$ $\quad$ <u>either</u> $\quad x \cdot y_i H = y_i H$

<u>or</u> $\quad \underbrace{x \cdot y_i H \neq x \cdot y_i H}$

if so, all diagonal entries of $\mathrm{Ind}(\rho)(x)$ are 0 in the (i,i)-block entry, and does not contribute to $tr(\mathrm{Ind}(\rho)(x))$

Then $(y_i^{-1} \cdot x \cdot y_i) H = H$

<u>and</u> $\quad Tr\left(\underbrace{\mathrm{Ind}(\rho)(x)\Big|_{k[y_i H]}} \right)$

$= \chi_\rho(x) \quad \boxed{}$ $\quad k[y_i H]$

$k[y_i H]$

<u>Exer</u> $\quad$ Write down an explicit expression of $Tr(\mathrm{Ind}(\rho)(x))$ following the above discussion.