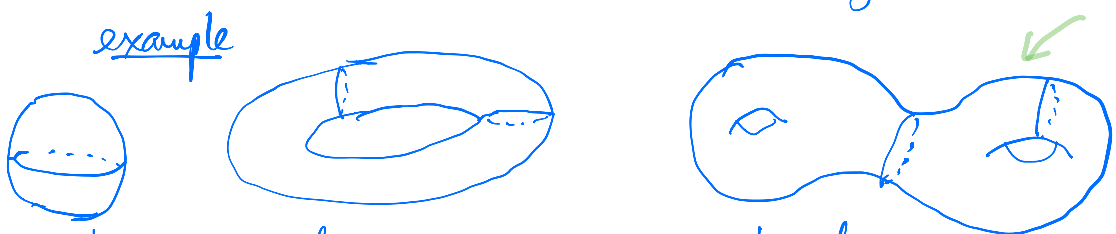
Categories and functors

Example Homology theories and cohomology theories.

Algebraic topology:

- Find "invariants" of geometric objects which can be used to distinguish them.

example



a sphere   a torus   a closed orientable surface in $\mathbb{R}^3$ of "genus" 2

These 3 orientable surfaces are not isotopic (can be continuous deformed to each other)

First answers - Bettie numbers.
invariant: a sequence of numbers.

e.g. Euler characteristic of oriented surfaces
"triangulate the objects".

$$\chi(S)\ \#(\text{vertices}) - \#(\text{edges}) + \#(\text{triangles})$$

independent of triangulation

"Analyse Situ" natural/canonical

Emmy Noether: abelian groups attached to a geometric objects.

Homology
Cohomology
Homotopy groups, etc

examples of invariants

"Geometric bodies / objects" — Examples:

{Piecewise linear spaces} ⊆

{Topological spaces}
"∪|"
{compactly generated topological spaces}
"∪|"
{CW complexes}
"∪|"
{manifolds (possibly with boundary)}
"∪|"
{real analytic vector spaces}

singular cohomology → $H^{\bullet}_{sing}$, $H_{\bullet}^{sing}$ ← singular homology

{Topological spaces} ⟶ {abelian groups}

↑ (from {CW complexes} to {Topological spaces}) ↑ = (on the right)

{CW complexes}
∪|
{simplicial complexes}

first defined

$H^{\bullet}$, $H_{\bullet}$ ⟶ {abelian groups}

(defined combinatorially)

{Topological Spaces} $\xrightarrow{H^i(\ , \mathbb{Z}/2\mathbb{Z})}$ {abelian groups killed by 2}

$\Downarrow Sq^k$ — natural transformation

$\xrightarrow{H^{i+k}(\ ; \mathbb{Z}/2\mathbb{Z})}$

"Cohomology operations".

Many other such functors: complex cobordism, K-theory

Classifying spaces

- cpx vector bundle on topological spaces.

E ← complex vector bundle.

$\downarrow \pi$

X ← top space

i.e. $\forall x \in X$ $\exists U \ni x$ open

and $\pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^r$

and transitions maps are continuous, preserving the $\mathbb{C}$-linear structure

General question: How to "classify" all cpx vector bundles of a given rk r over arbitrary "good" top spaces

$\mathcal{F}_r : X \rightsquigarrow$ (isomorphism classes of vector bundles over X)

↑ {"reasonable" topological spaces} ↑ (Sets)

Answer: There exists a topological space BU s.t.

$$\mathcal{F}_r(X) = [X, BU] = \{\text{continuous maps } X \to BU\} / \text{homotopy}$$

$$T = \{\text{top. spaces}\} \xrightarrow{\ \mathcal{F}_r\ } (\text{Sets})$$

$$T \downarrow \qquad \nearrow (\text{Sets})$$

$$Ho = \left\{\begin{array}{l}\text{top spaces} \\ + \text{ homotopy classes} \\ \text{of continuous maps}\end{array}\right\}$$

i.e. $\exists$ a cpx vector bundle $\mathcal{E}_{univ} \xrightarrow{\pi_{univ}} BU$ s.t. for every complex vector bundle $E \to X$, with $X \xrightarrow{f} BU$,

$\exists$ a continuous map $f : X \to BU$, unique up to homotopy, such that

$$f^* \mathcal{E} \simeq E.$$

i.e. $\mathcal{F}_r$ is a representable functor in the category Ho

Grothendieck group (associated to a additive semigroup)

<u>example</u>:

R: a ring

$\mathcal{P}_R = \left\{ \begin{array}{l} \text{finitely generated (presented)} \\ \text{projective } R\text{-modules} \end{array} \right\}$, a full subcategory of the category of R-modules.

$0 \to P' \to P \to P'' \to 0$ short exact, all f.g. proj. R-modules $\Rightarrow P \cong P' \oplus P''$

<u>def^n</u>

$$K(\mathcal{P}_R) = \frac{\left\{ \begin{array}{l} \text{the free abelian group} \\ \text{with generator } [P],\ P \in \mathcal{P}_R \end{array} \right\}}{\left(\begin{array}{l} \forall\ 0 \to P' \to P \to P'' \to 0 \text{ short exact} \\ [P] - [P'] - [P''] \end{array} \right)}$$

M_R = all finitely presented left R-modules

$\leadsto K(M_R)$

<u>Note</u>: when $R = O_F$ = the ring of all algebraic integers in a finite ext^n field $F/\mathbb{Q}$

then $K(M_{O_F}) = K(\mathcal{P}_{O_F})$ = the class group of F

Example: Classify all compact Riemann surfaces of genus two.

compact oriented ↗ connected 1-dim cpx manifold
- 2-dim surfaces with a conformal equiv. class of metrics

Classification / moduli problem

$\mathcal{M}_2$: every "reasonable" base space S,

$\rightsquigarrow$ { the isomorphic class of families $C \to S$ of compact Riemann surfaces of genus 2 }

Ans. / Phenomenon. $\exists$ a good answer

rigidified version $\rightsquigarrow$ Teichmuller spaces (of genus 2)

less rigid version $\rightsquigarrow$ $\mathcal{M}_2$ is represented by a complex algebraic variety of dimension 3

$\mathcal{M}_2$ is a representable functor in the category of { complex analytic varieties / (complex) algebraic varieties }