Tensor algebras, etc.

R: a commutative ring.

M: an R-module

$\rightsquigarrow$ $T^\bullet M = \bigoplus_{m\geq 0} \otimes_R^m M = R \oplus M \oplus (M\otimes M) \oplus \cdots$

an $\mathbb{N}$-graded ring

a "free associative R-algebra"

$$\mathrm{Hom}_{R,\text{ring}}(T^\bullet M, A) \xrightarrow[\text{bij.}]{\sim} \mathrm{Hom}_{R\text{-mod}}(M, A)$$

associative R-algebra

$R \xrightarrow{\text{ring homom}} Z(A) \to A$

Exer. Reformulate this in terms of adjoint functors.

<u>Ref</u> MacLane, Categories for working Mathematicians.

example: free associative R-alg with generator S

a set

$$T_R^\bullet\Big(\bigoplus_{s\in S} R\cdot s\Big) \text{"="} R\langle x_s\rangle_{s\in S}$$

non-commutative (but associative) polynomial ring with generators S

$\exists$ a canonical $\cong$ isom.

$= \left\{\begin{matrix}\text{formal R-linear combinations}\\ \text{of words in the alphabet set } S\end{matrix}\right\}$

$$R[x_s]_{s\in S} = T_R^\bullet\Big(\bigoplus_{s\in S} R\cdot s\Big)\Big/ \text{the ideal generated by } \{s_1s_2 - s_2s_1 : s_1, s_2 \in S\}$$

(standard polynomial ring with variables indexed by S)

$\exists$ a canonical isom.

Examples of non-commutative rings / ($\mathbb{R}$-algebras)

matrix algebras (and their subring/algebras)

On $C^\infty(\mathbb{R};\mathbb{C})$ — have 2 kinds of operations

1) multiply by a smooth function
2) $\frac{d^n}{dx^n}$

$\rightsquigarrow$ The generate the ring

$$\left\{\begin{array}{l}\text{all linear differential} \\ \text{operators with coeff. in } C^\infty(\mathbb{R};\mathbb{C})\end{array}\right\}$$

operating on $C^\infty(\mathbb{R};\mathbb{C})$

$$\sum_{\text{finite}} f_n(x)\frac{d^n}{dx^n}$$

the Weyl algebra $(n=1)$

$$\cup\!/ \qquad\qquad //$$

$$\left\{\begin{array}{l}\mathbb{C}\text{-subalg. generated} \\ \text{by } x, \frac{d}{dx}\end{array}\right\} \qquad \mathbb{C}\left[x, \frac{d}{dx}\right]$$

$$//$$

$$\mathbb{C}\langle x, y\rangle / (yx - xy - 1)$$

Symmetric tensor product:

R = a commutative ring

M: an R-module

$m \in \mathbb{N}$

$$S^m M \underset{\text{def}}{:=} \underbrace{M \otimes_R M \otimes_R \cdots \otimes_R M}_{m \text{ times}} \Big/ \left(\begin{array}{l}\text{the } R\text{-submodule generated} \\ \text{by } x_1 \otimes \cdots \otimes x_m - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)} \\ x_1, \ldots x_m \in M \quad \sigma \in S_m\end{array}\right)$$

the m-th symmetric product of M over R.

$$S_R^\bullet(M) \underset{\text{defn}}{=} \bigoplus_{m\in\mathbb{N}} S_R^m M .$$ ← a symmetric R-alg.

the symmetric R-algebra gen. by M

<u>Exer.</u> Formulate the universal properties for $S^m M$ and $S^\bullet(M)$, in terms of adjoint functors.

<u>Exer</u>

$$S_R^\bullet\Big(\bigoplus_{t\in T} R\cdot t\Big) = R[x_t]_{t\in T}$$

a set

$$T_R^\bullet(M) \supseteq \mathcal{S}(T_R^\bullet M)$$

$$= \bigoplus_{r\in\mathbb{N}} \underbrace{\left(\begin{array}{l}\text{symmetric elements}\\ \text{in } T_R^r(M)\end{array}\right)}_{\parallel \atop (T_R^r(M))^{S_r}}$$

Have natural action of S_r operating R-linearly on $T_R^r(M)$

<u>Question</u>: Are $S_R^\bullet(M)$ and $\mathcal{S}(T_R^\bullet M)$ "the same"?

has a natural structure as an R-algebra

?

$r=3$ $\quad M \otimes_R M \otimes_R M \ni \sum_{\text{finite}} a_i \underbrace{x_i \otimes y_i \otimes z_i}_{\substack{\parallel \\ \text{the image of} \\ (x_i, y_i, z_i) \text{ under} \\ \text{the canonical/natural map} \\ M \times M \times M \longrightarrow M \otimes_R M \otimes_R M}}$ $\quad x_i, y_i, z_i \in M$, $a_i \in R$

$S_3 \ni (12)$: $x_i \otimes y_i \otimes z_i \to y_i \otimes x_i \otimes z_i$

$\rightsquigarrow \underbrace{(M \otimes_R M \otimes_R M)^{S_3}}_{\substack{\text{symmetric 3-tensors} \\ \text{in } M^{\otimes 3}}} \underset{R\text{-submodule}}{\subseteq} M \otimes_R M \otimes_R M$

Exer: Assume M is a free R-module of finite rank n. (e.g. if $R=$ a field)

1) What is $(M^{\otimes r})^{S_r}$? Is it a free R-submodule? If so, what is its rank?

2) Same questions for $S^r(M)$.

e.g. Say $M = Rx_1 \oplus \cdots \oplus Rx_n$ free-R-module.

$$T^2_R(M) = M \otimes_R M = \left(\bigoplus_i R\cdot x_i\right) \otimes_R \left(\bigoplus_j R\cdot x_j\right)$$
$$= \bigoplus_{i,j} (Rx_i \otimes Rx_j) = \bigoplus_{1\le i,j\le n} R\cdot(x_i \otimes x_j)$$

a free R-module of rank n^2

symmetric 2-tensors

$$\mathcal{S}(T^2_R M) = \bigoplus_{1\le i<j\le n} R\cdot(x_i\otimes x_j + x_j \otimes x_i) \quad \bigoplus_{1\le i\le n} R\cdot(x_i \otimes x_i)$$

rank $= \frac{n(n+1)}{2}$

$$S^2_R(M) = \underbrace{T^2_R(M)}_{\substack{\| \\ \sum_{1\le i,j\le n} R\cdot(x_i\otimes x_j)}} \Big/ \big(x_i\otimes x_j - x_j\otimes x_i\big)$$

$$= \bigoplus_{1\le i\le j\le n} R\cdot(x_i\cdot x_j)$$

the image of $x_i\otimes x_j$ in $S^2_R(M)$

a free R-module of rank $\frac{n(n+1)}{2}$

Exterior / Alternating product

$$\Lambda^r_R M \overset{\text{def}}{=} \otimes^r_R M \Big/ \begin{pmatrix} R\text{-submodule generated by} \\ (x_1\otimes\cdots\otimes x_r) \\ \text{s.t. } \exists\, 1\le i\neq j\le r \text{ with } x_i = x_j \end{pmatrix}$$

R ← comm.

M ← R-module

$\leadsto$ $\otimes^r M \longrightarrow \Lambda^r M$ is R-multilinear and alternating

$$\Lambda^{\cdot}_R M \overset{\text{def}}{=} \bigoplus_{r\in\mathbb{N}} \Lambda^r_R M$$

↖ an R-algebra

<u>Exer</u>: $Sk(T^{\cdot}_R M) = \begin{pmatrix}\text{skew-symmetric tensors} \\ \text{in } T^{\cdot}_R M\end{pmatrix}$

$$Sk(T^{\cdot}_R M) = \bigoplus_{r\in\mathbb{N}} Sk(\otimes^r_R M)$$

$\cup$

$$Sk(\otimes^r_R M) = \left\{ y\in\otimes^r_R M \;\middle|\; \begin{array}{l}\sigma(y) = \mathrm{sgn}(\sigma)\cdot y \\ \forall \sigma\in S_r\end{array} \right\}$$

Is a relation between $\mathcal{SK}(T_R^{\cdot} M)$ and $\Lambda_R^{\cdot} M$?

Suppose M is a free R-module.

(a) Are $\Lambda_R^r M$ and $\mathcal{SK}(\otimes_R^r M)$ free?

(b) If so, what are their ranks?

(c) Are $\mathcal{SK}(T_R^{\cdot} M)$ and $\Lambda_R^{\cdot} M$ canonically isomorphic?

<u>Questions</u>: Let M be a free R-module, R: comm. ring.
<u>Exer.</u> of finite rank

$r \in \mathbb{N}$

$\rightsquigarrow \quad M^\vee = \mathrm{Hom}_R(M, R)$

1) $\bigotimes_R^r M, \quad \otimes_R^r (M^\vee), \quad (\otimes_R^r M)^\vee, \quad (\otimes_R^r M^\vee)^\vee$

2) $S_R^r M, \quad S_R^r M^\vee, \quad (S_R^r M)^\vee, \quad (S_R^r M^\vee)^\vee$

$\mathcal{S}(\otimes_R^r M), \quad \mathcal{S}(\otimes_R^r M^\vee)$, etc.

3) $\Lambda_R^r M, \quad \Lambda_R^r M^\vee, \quad (\Lambda_R^r M)^\vee$

$\mathcal{SK}(\otimes_R^r M), \quad \mathcal{SK}(\otimes_R^r M^\vee)$, etc

Any relation among them ??