Localization, prime ideals, maximal ideals, Zariski topology on Spec(R).

R: a commutative ring.

Def: 1) A <u>prime ideal</u> of R is an ideal $P \subsetneq R$ such that R/P is an integral domain.

2) A <u>maximal ideal</u> of R is an ideal $M \subsetneq R$ such that R/M is a <u>field</u>.

(<u>equiv</u>: The only ideals I in R st $M \subseteq I \subseteq R$ are M and R)

<u>Rmk</u> maximal ideals are prime ideals

<u>Recall</u>: (1) Every commu. ring has at least one max ideal. (<u>Exer.</u> Prove this statement using Zorn's lemma)

$\Longleftrightarrow$ $\forall$ ideal $I \subsetneq R$
$\exists$ a maximal ideal M of R which contains I

<u>Cor</u>: Existence of prime ideals in any given commutative ring.

Suppose S is a multiplicative subset $S \subset R \setminus \{0\}$, $1 \in S$

Let $h: R \longrightarrow S^{-1}R$ be the natural ring homomorphism.

<u>Lemma</u>: There exists a natural bijection

$$\left\{\begin{matrix}\text{prime ideals of } R \\ \text{which do not meet } S\end{matrix}\right\} \underset{\zeta^{-1}}{\overset{\zeta}{\rightleftarrows}} \left\{\begin{matrix}\text{prime ideals} \\ \text{of } S^{-1}R\end{matrix}\right\}$$

pf: ζ^{-1}: a prime ideal Q in $S^{-1}R \mapsto Q \cap R$
$:= h^{-1}(Q)$

ζ: a prime ideal P in R s.t. $P \cap S = \phi$ $\mapsto P \cdot S^{-1}R$
$:= h(P) \cdot S^{-1}R$
$\overset{\text{def}}{=}$ the ideal of $S^{-1}R$ gen. by $h(P)$.

(Exer.)

q.e.d.

Question: $R = \mathbb{Z}$, $S = \{ n \in \mathbb{Z} \mid \gcd(n, p) \}$
p: a prime number.

Find all prime ideals of $S^{-1}R$.

$$S^{-1}R = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{l} b, a \in \mathbb{Z} \\ b\mathbb{Z} + p\mathbb{Z} = 1 \end{array} \right\} = \underbrace{\mathbb{Z}_{(p)}}_{\uparrow\, p\mathbb{Z}}$$

$$S = \mathbb{Z} \setminus \underbrace{p\mathbb{Z}}_{\text{a prime ideal}}$$

a subring of $\mathbb{Q}$, hence an integral domain.

i.e. (0) is a prime ideal of $\mathbb{Z}_{(p)}$

$p \cdot \mathbb{Z}_{(p)}$ is a maximal ideal

$$\mathbb{Z}_{(p)} / p \cdot \mathbb{Z}_{(p)} \cong S^{-1} \cdot \underbrace{(\mathbb{Z}/p\mathbb{Z})}_{\shortparallel\; \mathbb{F}_p} = \mathbb{F}_p$$

i.e. $p\mathbb{Z}_p$ is a maximal ideal of $\mathbb{Z}_{(p)}$.

Lemma $\Rightarrow$ $\mathbb{Z}_{(p)}$ has only two prime ideals
(0), $p\mathbb{Z}_{(p)}$

In particular, $\mathbb{Z}_{(p)}$ is a commutative local ring

def^n: A commutative ring R is local if R has exactly one maximal ideal.

Lemma: Let R be a commutative ring. Let P be a prime ideal of R. Then $\underbrace{(R \setminus P)^{-1} R}_{\overset{!!}{R_P}}$ is a local ring, with $P \cdot R_P$ as the unique maximal ideal of R_P.

(Exer.)

Def: Let R be a commutative ring.

1) Let $\mathrm{Spec}(R) := \{P : \text{prime ideals of } R\}$
(spectrum of R)

2) Define the Zariski topology on $\mathrm{Spec}(R)$,
Oscar Zariski.

as follows:

(a) A subset T of $\mathrm{Spec}(R)$ is closed iff $\exists$ an ideal $I \subseteq R$ s.t
$$T = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$$
(exer show this defines a topology on $\mathrm{Spec}(R)$)

(b) Subsets of the form
$$h_f^{-1}(\mathrm{Spec}((f^{\mathbb{N}})^{-1}R)) = U_f = \{P \in \mathrm{Spec}(R) \mid P \not\ni f\}$$
where $f \in R$, f not nilpotent
$$f^{\mathbb{N}} = \{f^n \mid n \in \mathbb{N}\}$$
$$h_f : R \longrightarrow \underbrace{(f^{\mathbb{N}})^{-1}R}_{\substack{\| \\ R_f}}$$
forms a basis of open neighborhoods of Spec(R).

(<u>Exer.</u>)

<u>Idea</u> / Phenomenon!) There is a sheaf of commutative rings on Spec(R), called the structural sheaf on Spec(R), often denoted by $\mathcal{O}_{\mathrm{Spec}(R)}$, characterized by the property that
$$\Gamma(\underbrace{\mathrm{Spec}(R_f)}_{\substack{\uparrow \\ \text{an open subset} \\ \text{of } \mathrm{Spec}(R)}}, \underbrace{\mathcal{O}_{\mathrm{Spec}(R)}}_{\substack{\uparrow \\ \text{a sheaf on } \mathrm{Spec}(R)}}) = R_f \qquad \forall \underset{\text{non-nilp}}{f} \in R$$

In particular,
$$\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R$$

2) Every R module M defines a sheaf $\widetilde{M}$ of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules on $\mathrm{Spec}(R)$, characterized by

$$\Gamma(\mathrm{Spec}(R_f), \widetilde{M}) = M_f := (f^N)^{-1} M$$

$\forall f \in R$ (non-nilp.).

<u>Example</u>: $R = \mathbb{C}[x, y]$

$\mathrm{Spec}(\mathbb{C}[x,y])$

(Hilbert Nullstellensatz: $\mathrm{Max}(R) = \mathbb{C}^2$)

$\mathrm{Max}(R) = \begin{Bmatrix} \text{maximal} \\ \text{ideals of } R \end{Bmatrix}$, $\mathbb{C}^2 \ni (z,w)$

$$\mathfrak{m}_{(z,w)} := \begin{Bmatrix} f(x,y) \in \mathbb{C}[x,y] \\ \text{s.t. } f(z,w) = 0 \end{Bmatrix}$$

<u>Fact</u>: There are 3 kinds of prime ideals in $\mathbb{C}[x,y]$.

(0), $\underbrace{\mathfrak{m}_{(z,w)} = (x-z)\mathbb{C}[x,y] + (y-w)\mathbb{C}[x,y], \quad (z,w) \in \mathbb{C}^2}_{\text{maximal ideals of } \mathbb{C}[x,y]}$

the generic pt.

$f(x,y)\,\mathbb{C}[x,y]$, where $f(x,y)$ is an irreducible elt. of $\mathbb{C}[x,y]$.

The Zariski topology on $\mathrm{Spec}\,\mathbb{C}[x,y]$ has many open subsets:

$U_f := \{ P \in \mathrm{Spec}(\mathbb{C}[x,y]) \mid P \not\ni f \}$ $\quad f \in R$

$\mathrm{Spec}(\mathbb{C}[x,y]) \setminus U_f = \{ P \in \mathrm{Spec}(\mathbb{C}[x,y]) \mid P \ni f \}$

$\quad \| \qquad \overset{\text{can. bij}}{=} \; \mathrm{Spec}\left(\mathbb{C}[x,y] / f \cdot \mathbb{C}[x,y] \right)$

$V(f\,\mathbb{C}[x,y])$

Given $\underset{0^{\times}}{f} \in \mathbb{C}[x,y] \rightsquigarrow \{ (a,b) \in \mathbb{C}^2 \mid f(a,b) = 0 \}$ $\;\overset{V_f^{(0)}}{//}$

$\cong\!\!/$ "the affine algebraic curve on $\mathbb{C}^2$"

$V(f)$ $= $ the ~~zero~~ zero locus of $f(x,y)$

$V(f) := V_f^{(0)} \cup \{ g\,\mathbb{C}[x,y] \mid g \text{ is an irreducible factor of } f \}$

$f = a \cdot g_1^{e_1} \cdots g_r^{e_r} \quad g_i \text{: irred}$, $\quad a \in \mathbb{C}^{\times}$

$= V_f^{(0)} \cup \{ \text{generic pts of irred. components of the zero locus of } f \}$

$\Gamma(U_f, \mathcal{O}_{\mathrm{Spec}(R)}) =$ rational functions which are regular (i.e. no poles) on U_f

$= \mathbb{C}[x,y][\tfrac{1}{f}] = (f^n)^{-1}\mathbb{C}[x,y].$