Review: character tables + tips for "guessing" character tables from partial information

$\mathbb{C}$ G = finite group. $\underset{\substack{\Downarrow \\ 1_G}}{\overset{\substack{1_G \\ \|}}{C_1}}, \ldots, \underset{\substack{\Downarrow \\ z_r}}{C_r}$ conjugacy classes of G

$\mathbb{1} = \chi_1, \ldots, \chi_r$ irred. cpx characters of G $\quad \#(C_j) = c_j \quad j = 1, \ldots, r$

	1 C_1	$\cdots$	c_j C_j		c_r C_r
$\mathbb{1} = \chi_1$	1		1		1
$\vdots$					
χ_α	$\chi_\alpha(1)$		$\chi_\alpha(z_j)$		
$\vdots$					
χ_r	$\chi_r(1)$				

orthogonality relations:

$$\frac{1}{\#(G)} \sum_{j=1}^{r} c_j \, \chi_\alpha(z_j) \cdot \underbrace{\chi_\beta(z_j^{-1})}_{\substack{\| \\ \overline{\chi_\beta(z_j)}}} = \delta_{\alpha\beta}$$

$$\frac{c_j}{\#(G)} \sum \chi_\alpha(z_i) \quad \chi_\alpha(z_j^{-1}) = \delta_{ij}$$

Remarks (1) $\chi_\alpha(z_j)$ is a sum of $\chi_\alpha(1)$ roots of 1, $\mu_1, \ldots, \mu_{\chi_\alpha(1)}$.

$\mu_i^{\operatorname{ord}(z_j)} = 1 \quad \forall\, i = 1, \ldots, \chi_\alpha(1)$

Consequently: (1a) $\chi_\alpha(z_j)$ is an algebraic integer in $\mathbb{Q}(\mu_{\mathrm{ord}(z_j)})$

$$\cap\!\!\!\!\mid \quad \underbrace{\mathbb{Q}(\mu_{\#G})}_{K}$$

(1b) $|\chi_\alpha(z_j)| \leq \chi_\alpha(1)$

(and also for all Galois conjugates of $\chi_\alpha(z_j)$ in K.

and if $=$ holds, then

$$\rho_\alpha(z_j) \in \underbrace{\mathbb{C}^\times}_{\text{a root of } 1.} \cdot \mathrm{Id}_{V_\alpha}$$

(2) $\forall z_j$, $\overline{\chi_\alpha(z_j)} = \chi_\alpha(z_j^{-1})$

or $\forall$ any element in the conjugacy class of z_j^{-1}

$= \chi_\beta(z_j)$,

$\rho_\beta \triangleq$ the contragradient repr. of α

(3) Let $m \in \mathbb{Z}$, consider z_j^m

$C_j \ni z_j \longmapsto z_j^m$

$\ \uparrow$

$C_{k(j,m)}$

$\rightsquigarrow$

$$C_j \longmapsto C_{k(j,m)}$$

depends only on j, m

$\forall \alpha$

$$\chi_\alpha(z_j) = \lambda_1 + \cdots + \lambda_{\chi_\alpha(1)}$$

$$\chi_\alpha(z_j^m) = \lambda_1^m + \cdots + \lambda_{\chi_\alpha(1)}^m$$

Conclusion: (Given the structure of the group)

$\forall z_j$, Fix α
C_j

Look at $\chi_\alpha(z_{k(j,m)})$ $C_{k(j,m)}$ $m \in \mathbb{Z}$

$\rightsquigarrow$ They determine all Newton polynomials of the eigenvalues of $\rho_\alpha(z_j)$ (only need $m = 0, \dots, \mathrm{ord}(z_j)-1$)

$\rightsquigarrow$ get the conjugacy class of $\rho_\alpha(z_j)$ in $GL_{\chi(1)}(\mathbb{C})$

In particular you can "read off" the order of an element of G from the character table

("Even if you don't know the structure of G")

(3) Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, Galois over $\mathbb{Q}$

Pick an automorphism $\tilde{\sigma}$ of $\mathbb{C}$ such that

$$\tilde{\sigma}\big|_K = \sigma$$

Consider $\rho_\alpha : G \to GL_n(\mathbb{C})$ irred repr., $n = \chi_\alpha(1)$

$$\rightsquigarrow \quad \tilde{\sigma}\rho_\alpha : \begin{array}{ccc} G & \longrightarrow & GL_n(\mathbb{C}) \\ \Psi & & \Psi \\ x & \longmapsto & \tilde{\sigma}(\rho_\alpha(x)) \end{array} \qquad \begin{array}{c} \tilde{\sigma} \\ GL_n(\mathbb{C}) \\ \downarrow \\ GL_n(\mathbb{C}) \end{array}$$

is an (irred.) repr. of G

$$\boxed{\chi_{\tilde{\sigma}(\rho_\alpha)}(x) = \sigma(\chi_\alpha(x))}$$

Let $n = \# G$

Fact: $Ar := \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \xrightarrow{\simeq} (\mathbb{Z}/n\mathbb{Z})^\times$

$\forall \bar{i} \in (\mathbb{Z}/n\mathbb{Z})^\times$, have
$$\begin{array}{ccc} \mu_n & \longrightarrow & \mu_n \\ \Psi & & \Psi \\ \zeta & \longmapsto & \zeta^{\bar{i}} \end{array}$$

i.e. $(\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\simeq} \mathrm{Aut}(\mu_n)$

Every field autom. of $\mathbb{Q}(\mu_n)$ induces an automorphism of the group $\mu_n(\mathbb{C}) \subseteq \mathbb{C}^\times$

$\forall \bar{i} \in (\mathbb{Z}/n\mathbb{Z})^\times$, $\forall \alpha = 1, \dots, r$

The function
$$\begin{array}{ccc} x & \longmapsto & Ar(\bar{i})(\chi_\alpha(x)) \\ \cap & & \\ G & & \end{array}$$
where $Ar(\bar{i}) \in \mathrm{Aut}(\mathbb{Q}(\mu_n))$ and $\chi_\alpha(x) \in \mathbb{Q}(\mu_n)$

is an irreducible character of G!

(Have seen one example before: $i = -1 \pmod n$)

Linear algebra constructions

(V_1, ρ_1), (V_2, ρ_2) are representations of G

$\rightsquigarrow$ $(V_1, \rho_1) \oplus (V_2, \rho_2)$

$(V_1 \otimes V_2, \rho_1 \otimes \rho_2)$

every element $x \in G$ operates on $V_1 \otimes V_2$ by $\rho_1(x) \otimes \rho_2(x)$

$$\boxed{\begin{aligned} &tr(\rho_1(x) \otimes \rho_2(x)) \\ &= tr(\rho_1(x)) \cdot tr(\rho_2(x)) \end{aligned}}$$

$V_1^\vee \otimes V_2$
$\|$
$Hom_{\mathbb{C}}(V_1, V_2)$

every element $x \in G$ operates on $Hom_{\mathbb{C}}(V, W)$ as

$$\underset{Hom_{\mathbb{C}}(V,W)}{h} \longmapsto (v \longmapsto \rho_2(x) \circ h \circ (\rho_1(x^{-1})))$$

(V, ρ) repr.

Special case: $Hom_{\mathbb{C}}(V, \mathbb{C})$

$\rightsquigarrow$ the contragradient repr.

(V, ρ) : repr. of G

$\rightsquigarrow \quad V^{\oplus m}, \quad \underbrace{V \otimes \cdots \otimes V}_{n \text{ copies}} =: V^{\otimes n}$

$\check{V}$ = contragradient repr. of V.

$(V^{\vee})^{\otimes n}$

$S^2(V, \rho) \subseteq (V, \rho) \otimes (V, \rho)$

$\rightsquigarrow \quad S^m(V, \rho)$

$\rightsquigarrow \quad \Lambda^m(V, \rho)$

Example S_4 :

	1 (1)	6 $(1,2)$	8 (123)	3 $(12)(34)$	6 (1234)
$\mathbb{1} = \chi_1$	1	1	1	1	1
$sgn = \chi_2$	1	-1	1	1	-1
$\chi_U = \chi_3$	3	1	0	-1	-1
$sgn \cdot \chi_U = \chi_4$	3	-1	0	-1	1
Find an irred & 2-dim repr. of S_4 $\longrightarrow$ χ_5	2	0	-1	2	0

$(\chi_U, \chi_U) = \frac{1}{24} \cdot (9 + 6 + 3 + 6) = 1$

χ_U	3	1	0	-1	-1
χ_{perm}	4	2	1	0	0

S_4 has a permutation repr. ρ_{perm} on $\mathbb{C}^4$

ρ_{perm} has a 1-trivial repr

$$\chi_{perm}(\underset{\substack{\uparrow \\ \text{permutation} \\ \text{in } S_n}}{\sigma}) = \#\{ i \in \{1,2,3,4\} \mid \sigma(i) = i \}$$

$$V_{perm} \cong \mathbb{1} \oplus U$$

(V,ρ) irred repr. of G

(W,ν) 1-dim^{l} repr. of G $\quad (W^\vee, \nu^\vee)$ irred.

$$\underline{V \otimes W} \overset{\text{if}}{\cong} U_1 \oplus U_2$$

$$\Longrightarrow \underbrace{(V \otimes W) \otimes W^\vee}_{\substack{\text{H3} \\ V}} = (U_1 \otimes W^\vee) \oplus (U_2 \otimes W^\vee)$$

one of them is 0

<u>so</u> either $U_1 = 0$

or $U_2 = 0$.

every elt has finite order

Theorem (Burside) Every periodic subgroup of bounded order of $GL_n(\mathbb{C})$ is finite!

Pf: Given $G \subseteq GL_n(\mathbb{C})$ subgroup s.t. $x^m = I_n$ $\forall x \in G$, $m \geq 1$

Assume $(\mathbb{C}^n, \text{std})|_G = \text{irred}$

Orthogonality relation of matrix coeff for $(\mathbb{C}^n \ \text{std})|_G$

$\Longrightarrow \exists\ n^2$ elts $x_1, \dots, x_{n^2} \in G$ s.t.

$$\underbrace{((i,j), \alpha)}_{1 \leq i,j \leq n} \longmapsto \Big(\underbrace{f_{ij}}_{\text{matrix coeff}}(x_\alpha)\Big)$$

is an "$n^2 \times n^2$" invertible matrix.

Have

$$(*) \qquad \underbrace{\chi(x_\alpha \cdot y)}_{\text{a sum of } n \text{ elts of } \mu_m(\mathbb{C})} = \sum_{1 \leq i,j \leq n} f_{ij}(x_\alpha)\, f_{ji}(y) \qquad \forall y \in G$$

a sum of n elts of $\mu_m(\mathbb{C})$ ⇝ only finitely many possibility for $\chi(x_\alpha \cdot y)$

a linear system of eq^n. in $f_{ji}(y)$

n^2 unknown, and n^2 equations (one for each x_α)

⇝ done in this case.

General case

$\exists$ a Jordan-Hölder series

$$\mathbb{C}^n \supseteq V_r \supsetneq V_{r-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0 = 0$$

s.t. V_{i+1}/V_i is irred. for

$i = 0, 1, \ldots, r-1$.

What we just showed:

$$\Longrightarrow \begin{pmatrix} \boxed{*_1} & & & \\ & \boxed{*_2} & & \\ & & \ddots & \\ & & & \boxed{*} \end{pmatrix} \supseteq \begin{pmatrix} \boxed{1} & ? & & \\ & \boxed{1} & \ddots & ? \\ & & \ddots & \\ & & & \boxed{1} \end{pmatrix}$$

Image of G in $(*_i)$ = finite

if $\neq 1_n$, $\Longrightarrow$ must have infinite order!

QED.