Categories and functors

Last Friday: moduli functors.

Categories

Functors : $F : \mathcal{C} \to \mathcal{D}$ covariant –

$$Ob(\mathcal{C}) \ni X \qquad F(X) \in Ob(\mathcal{D})$$

$$\begin{array}{ccc} X & & F(X) \\ f \downarrow & & \downarrow F(f) \\ Y & & F(Y) \end{array}$$

satisfying "obvious properties".

$$F(id_X) = id_{F(X)}$$

$$F(g \circ f) = F(g) \circ F(f)$$

Adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

functorial natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(G(X), Y) \overset{}{\underset{\alpha(X,Y)}{\longleftrightarrow}} \mathrm{Hom}_{\mathcal{D}}(X, F(Y))$$

$\forall \ X \in Ob(\mathcal{D})$

We say G is the left adjoint of F

and F is the right adjoint of G

Example: 1) free commutative R-algebra with a given set of free generators S, where R is a commutative algebra.

$R[X_s]_{s \in S}$ polynomial alg. over R with generator set S

$$R[X_s]_{s\in S} = \left\{ \sum_{\substack{I: S \to \mathbb{N} \\ \text{finite support}}} a_I \cdot \underbrace{\prod_{s \in S} X_s^{a_s}}_{\uparrow \text{ monomial}} \;\middle|\; a_I \in R \quad \forall I \right\}$$

$\left(R[X_s]_{s\in S},\ S\right)$ has the following universal property:

$\forall$ commutative R-algebra A.

have a natural / functorial bijection

$$\mathrm{Hom}_{\text{comm. } R\text{-alg}}\left(R[X_s]_{s\in S}, A\right) \longleftrightarrow \left(\text{functions from } S \text{ to } A\right) = \mathrm{Hom}_{\text{Sets}}(S, \text{the set underlying } A)$$

$$\begin{tikzcd}
\left(\begin{array}{c}\text{Commutative} \\ R\text{-alg.}\end{array}\right) \arrow[r, shift left, "\text{``stripping''}"] & (\text{Sets}) \arrow[l, shift left] \\
R[X_s]_{s\in S} & S \arrow[l, mapsto] \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

Tensor products

R: a ring, M: a right R-module

N: a left R-module

⇝ $M \otimes_R N$ = a $\mathbb{Z}$-module

characterizing property:

i) $M \longrightarrow M \otimes_R N$
$N \longrightarrow M \otimes_R N$

ii) $\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N \longrightarrow A)$ (A: $\mathbb{Z}$-module)

$\xleftrightarrow{\text{functorial bijection}} \left\{ f : M \times N \longrightarrow A \;\middle|\; \begin{array}{l} \mathbb{Z}\text{-bilinear} \\ f(m\cdot r, n) = f(m, rn) \\ \forall m \in M, \forall n \in N, \forall r \in R \end{array} \right\}$

Exer. Reformulate this in terms of adjoint functors

$(m_1+m_2, n) - (m_1, n) - (m_2, n)$

Construction

$M \otimes_R N \overset{\text{"def"}}{=\!=} \left\{ \begin{array}{l} \text{free abelian group} \\ \text{with basis } (m,n) \\ m \in M,\ n \in N \end{array} \right\} \Big/ \begin{array}{l} \text{the subgroup} \\ \text{generated by} \\ (m_1+m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1+n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn) \end{array}$

a left adjoint.

$m_1, m_2, m \in M$, $n_1, n_2, n \in N$, $r \in R$

Check: The required universal property holds!

Properties/Remark

natural isomorphic -le

1) $(M_1 \oplus M_2) \otimes_R N = (M_1 \otimes_R N) \oplus (M_1 \otimes_R N)$

$M \otimes_R (N_1 \oplus N_2) = (M \otimes_R N_1) \otimes (M \otimes_R N_2)$

2) $$(M/M') \otimes_R N = M \otimes_R N / \mathrm{Im}(M' \otimes_R N \to M \otimes_R N)$$

right R-submodule of M

abuse notation

$$= M \otimes_R N / M' \otimes_R N$$

$$M \otimes_R (N/N') = M \otimes_R N / \mathrm{Im}(M \otimes_R N' \to M \otimes_R N)$$

3) $R \otimes_R N = N$

nat. isom. to

$M \otimes_R R = M$

4) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of left R-modules, then

$$M' \otimes_R N \xrightarrow{\alpha} M \otimes_R N \xrightarrow[\text{surj.}]{\beta} M'' \otimes_R N \to 0$$

may not injective (exer.)

$\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$

is an exact sequence of $\mathbb{Z}$-modules "right exact in M"

Similarly: $\otimes$ is right exact in N.

<u>Remark</u>: (i) If $0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$ holds $\forall$ short exact sequence $0 \to M' \to M \to M'' \to 0$ R-modules.

Say that N is a <u>flat</u> left R-module.

(ii) In <u>general</u>: $\exists$ a long exact sequence.

may not be 0

$$\begin{tikzcd}
\mathrm{Tor}_1^R(M,N) \arrow[r] & \mathrm{Tor}_1^R(M'',N) \arrow[r] & M' \otimes_R N \arrow[r, "\alpha"] & M \otimes_R N \arrow[r, "\beta"] & M'' \otimes_R N \arrow[r] & 0 \\
\mathrm{Tor}_1^R(M',N) \arrow[u] & \arrow[l] & & & &
\end{tikzcd}$$

<u>Examples of</u> R s.t. the arrow "α" is always an injection?

e.g. When R is a <u>field</u>!

Or more general, when R is a division ring.

<u>possible</u> More structure on $M \otimes_R N$ $\qquad {}_S M_R \quad {}_R N$

1) If M has a structure as a left S-module for some ring S compatible with R, <u>i.e.</u> M is an (S,R)-bimodule

$$s(mr) = (s \cdot m) \cdot r \quad \forall s \in S \ \forall r \in R \ \forall m \in M$$

<u>then</u>: $M \otimes_R N$ has a natural structure as a <u>left</u> S-module

Similarly when N is an (R,S)-bimodule

In particular, if R is commutative, then $M \otimes_R N$ has a natural R-module str.

"<u>Associativity</u>" R, S : rings

$$M_R,\ {}_R N_S,\ {}_S P$$

$$(M \otimes_R N) \otimes_S P \xrightarrow{\simeq} M \otimes_R (N \otimes_S P)$$

canonical (functorial / natural

<u>Special case</u>: R : commutative

M: R-module   convention $\otimes^n M = R$

$$\bigoplus_{n \in \mathbb{N}} \otimes_R^n M =: T_R^\bullet M$$

an $\mathbb{N}$-graded R-algebra.   "free associative tensor algebra of M"

<u>Exer</u>. Find a universal property which an adjoint pair which characterizes $T_R^\bullet M$.