Elementary character theory (cpx linear repr. of finite groups)

Character table of a finite group.

$r = \#$ of conjugacy classes of G

$= \#$ irred $\mathbb{C}$-representations

any algebraically closed field k s.t. $1_k \cdot \#(G) \in k^\times$

Usually: $\operatorname{char}(k) = 0$

$\{1_G\} = C_1, \dots, C_i, \dots, C_r$ conjugacy classes

$z_i \in C_i$ $\qquad \#(C_i) = c_i$

$1 = \chi_1, \dots, \chi_r$ irred. characters

character table

$$\begin{array}{c|ccc} & C_1 & \cdots & C_r \\ \hline \chi_1 & & & \\ \vdots & & \chi_\alpha(z_i) & \\ \chi_r & & & \end{array}$$

Define: For any two functions $f_1, f_2 : G \to k$

define $\langle f_1, f_2 \rangle = \dfrac{1}{\#G} \sum_{t \in G} f_1(t)\, f_2(t^{-1})$

Notice: For a unitary repr. (V, ρ) of G.

$$\chi_V(t^{-1}) = \overline{\chi_V(t)} \quad \forall\, t \in G$$

$(V_1, \rho_1), \ldots, (V_r, \rho_r)$: irred. k-repr. of G

$\mathbb{C}$ mutually non-isom.

$\chi_\alpha(t) = \mathrm{Tr}_{V_\alpha}(t) \quad \forall t \in G$

Saw: $\langle \chi_\alpha, \chi_\beta \rangle = \delta_{\alpha,\beta}$

$\leadsto$ the irred. character generate an s-dim^l subspace of $Z(k[G])$

Will show: $s = r$

Here, we identified $k[G]$ with functions on G

$$\underset{\substack{\text{a function}\\ \text{on } G}}{(t \mapsto a_t)} \longleftrightarrow \sum_{t \in G} a_t [t]$$

Lemma. (1) Suppose $u = \sum_{t \in G} a_t [t] \in Z(k[G])$.

$k[G] \ni u \quad \longleftrightarrow \quad f \qquad f(t) = a_t \quad \forall t \in G$

Then u operates on (V_β, ρ_β) by $\beta = 1, \ldots, s$

by

$$\underbrace{\frac{1}{\chi_\beta(1)} \sum_{t \in G} a_t \cdot \chi_\beta(t)}_{\displaystyle \| \atop \displaystyle \frac{\#(G)}{\chi_\beta(1)} \langle f, \chi_{\beta^\vee} \rangle} \cdot \mathrm{Id}_{V_\beta}$$

where $\chi_{\beta^\vee}$ = character of the contragradient repr of (V_β, ρ_β)

Recall: $\forall$ finite dim^l repr. (V, ρ) of G.
Have its contragradient repr. $(V^\vee, \rho^\vee)$, where

$$V^\vee = \mathrm{Hom}_k(V, k)$$

$$\rho^\vee(x) = \underbrace{\rho(x^{-1})^\vee}_{\text{transpose of } \rho(x^{-1})} \quad \forall x \in G$$

(2) The element $\sigma_i = \sum_{y \in C_i} [y] \in Z(k[G])$

operates on the irred repr. (V_β, ρ_β) as

$$\underbrace{\frac{c_i \cdot \chi_\beta(z_i)}{\chi_\beta(1)}}_{\text{integral}/\mathbb{Z}} \cdot \mathrm{Id}_{V_\beta}$$

$\mathbb{Z}[G]$ | integral! $\mathbb{Z}$

1) $Z(\mathbb{Z}[G])$: a comm. algebra free of finite rk over $\mathbb{Z}$

2) σ_i satisfies a monic poly equation with integer coeff.

(3) $\forall \alpha = 1, \dots, s.$

$\sum_{y \in G} \chi_\alpha(y)\, [y]$ operates on V_β as

$$\frac{\# G}{\chi_\alpha(1)} \underbrace{\langle \chi_\alpha, \chi_{\beta^\vee} \rangle}_{\delta_{\alpha, \beta^\vee}} \cdot \mathrm{Id}_{V_\beta}$$

operates on V_β by linear combinations of the $\rho_\beta(\sigma_i)$ with coeff. in alg. integers

$\chi_\alpha(y)$ = a sum of roots of 1!

$\because \rho_\alpha(y)^{\#(G)} = \mathrm{Id}_{V_\alpha}$

Pf of (1): $\forall \beta = 1, \dots, s$

$$\underbrace{\sum_{y \in G} a_y [y]}_{\in Z(k[G])} \underset{\text{Schur's Lemma}}{\Longrightarrow} \rho_\beta\left(\sum_{y \in G} a_y [y]\right) = \overset{k}{\underset{\cup}{?}} \cdot \mathrm{Id}_{V_\beta}$$

compute: $$? \cdot \chi_\beta(1) = \sum_{y \in G} a_y \cdot \chi_\beta(y) = \#G \langle f, \chi_{\beta^\vee} \rangle$$

q.e.d.

k-linear repr. of G

equiv (1) left $k[G]$-module

(2) a linear left action of G, $G \times V \to V$

(3) a group homom $G \to GL_k(V)$

(4) a ring homom. $k[G] \to \mathrm{End}_k(V)$

$u \in Z(k[G]) \subseteq k[G]$

(1) $\Rightarrow$ (2): Given a left $k[G]$-module V.

define $$G \times V \xrightarrow{\mu} V$$
$$(x, v) \longmapsto [x] \cdot v$$

$k \supseteq \mathbb{Q}$ ($k = \mathbb{C}$)

Cor. (a) $\underbrace{\dfrac{c_i \chi_\beta(z_i)}{\chi_\beta(1)}}$ is an algebraic integer

$\forall$ conjugacy class C_i

$\forall$ irred. character χ_β

Exer: Show (a) $\Rightarrow$ (b) directly!

(b) $\chi_\beta(1) \mid \#(G)$ $\forall$ irred. character χ_β of G

$\dim(V_\beta)$ $\therefore$ $\dfrac{\#G}{\chi_\beta(1)}$ is an algebraic integer and $\in \mathbb{Q}$.

$\Rightarrow \in \mathbb{Z}$. q.e.d.

Note $\chi_\alpha(z_i)$ = the sum of $\overbrace{\text{all eigenvalues}}^{\text{all are in } \mu_{\#G}}$ of $\rho_\alpha(z_i) \in \mathrm{End}(V_\alpha)$

a typical elt of the character table of G

$\in \mathcal{O}_{\mathbb{Q}(\mu_{\#G})}$

$\|$

$\mathbb{Z}[\zeta_{\#G}]$

a primitive $\#(G)$-th root of 1.

\# of irred characters of G

Prop: $s = r$

\# of conjugacy class $= \dim Z(k[G])$

Pf: Know $s \leq r$ by Lemma 1

If $s \lneqq r$, then $\exists$ a non-zero element

$0 \neq u \in Z(k[G])$

which operates trivially on every irred. repr. of G.

That's absurd, because u operates nontrivially on $\mathbb{C}[G] \cong$ direct sum of (copies of) irred. repr. of G q.e.d.

Application — Burnside p,q theorem.

Theorem (Burnside) Let G be a finite group.

$a,b>0$

$n = |G| = p^a \cdot q^b$ $\quad p,q$: prime numbers, $p \neq q$, $a,b \in \mathbb{N}$.

$P =$ a p-Sylow . $c = [G : Z_G(h)]$

$P \supseteq Z(P) \ni h \neq 1$

Then G is solvable!

$= \mathbb{Q}(\mu_n) \subseteq \mathbb{C}$

Let $K = \mathbb{Q}(\mu_{\#G})$

$\supseteq \mathcal{O}_K =$ ring of integers in K $= \mathbb{Z}[\mu_{\#G}]$

Pf: 1. (Locate a suitable irred. character)

$\exists$ a irred. character $\chi_{\alpha_0} \neq \mathbb{1}$

s.t. $\boxed{*}$ $\begin{cases} \chi_{\alpha_0}(h) \notin q \cdot \mathcal{O}_K \\ \text{and} \\ \chi_{\alpha_0}(1) \not\equiv 0 \pmod q \end{cases}$

$\because$ $0 = \sum_\alpha \chi_\alpha(1)\, \chi_\alpha(h) = 1 + \underbrace{\sum_{\alpha \geq 2} \chi_\alpha(1)\, \chi_\alpha(h)}$

$\exists \alpha_0$ s.t. $\chi_\alpha(1) \cdot \chi_\alpha(h) \notin q \cdot \mathcal{O}_K$.

2. (Use integrality)

$\because$ $\gcd(c, \chi_{\alpha_0}(1)) = 1$

and $\underbrace{c \cdot \dfrac{\chi_{\alpha_0}(h)}{\chi_{\alpha_0}(1)} \in \mathcal{O}_K}$ Cor before

$\Longrightarrow$ $\dfrac{\chi_{\alpha_0}(h)}{\chi_{\alpha_0}(1)} \in \mathcal{O}_K$

Notice: This is an algebraic integer

and: all of its complex absolute values are ≤ 1

($\because$ $\chi_{\alpha_0}(h)$ is a sum of $\chi_{\alpha_0}(1)$ roots of 1)

On the other hand: $\forall$ non-zero integral element x in K $\leftarrow$ cyclotomic field,

The product of all complex absolute values of x are ≥ 1

($\because$ product formula)

or: $$\prod_{\sigma \in Aut(G)} \left| \sqrt[con]{\sigma(x)} \right| \in \mathbb{Z}_{\geq 1}$$

$\Rightarrow$ All complex abs. values of $\chi_{\alpha_0}(h)$ are 1

$\Rightarrow$ All eigenvalues of $\rho_{\alpha_0}(h)$ are equal, i.e. $\boxed{\rho_{\alpha_0}(h) \in \mathbb{C}^{\times} \cdot Id_{V_{\alpha_0}}}$

$\Longrightarrow$ either $\rho_{\alpha_0}^{-1}(\mathbb{C}^{\times} \cdot Id_{V_{\alpha_0}})$ is a non-trivial proper normal subgroup of G ($\chi_{\alpha_0}(1) \neq 1$)

Exer.:

or $\quad \chi_{\alpha_0}(1) = 1$ and $\rho_{\alpha_0}^{-1}(1)$ is a non-trivial proper normal subgroup of G.

Finish by induction.