Localization $\quad R$: comm. ring

$S \subseteq R \setminus \{0\}$

multiplicative, $1 \in S$

$\rightsquigarrow$ Construct $S^{-1}R$: a commutative ring and a ring homom. $R \xrightarrow{\alpha} S^{-1}R$

s.t. $\forall$ commutative ring T and every ring homom $\beta: R \to T$

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & T \\ S^{-1}R \arrow[ur, dotted, "\exists!\,\gamma"'] & \end{tikzcd}$$

s.t. $\beta(s) \in T^{\times}$ $\forall s \in S$

then $\exists!$ ring homom $\gamma: S^{-1}R \to T$

s.t. $\gamma \circ \alpha = \beta$.

(Exer. Formulate this property as an adjoint pair
i.e. using adjoint functors.)

Also: $\forall$ R-module M,
have an $S^{-1}R$-module $S^{-1}M$:

Construction $S^{-1}M = S \times M \big/ \sim$

$s_1^{-1} \cdot m_1$

$(s_1, m_1) \sim (s_2, m_2)$
$\iff \exists s_3 \in S$ s.t
$s_3(s_2 m_1 - s_1 m_2) = 0$

Exer: Formulate a universal property which characterizes $S^{-1}M$ ← localization of modules

Facts about localizations

1. $S^{-1}M$ has a natural structure as a module over $S^{-1}R$.

$$[(t, x)] \cdot [(s, m)] = [(ts, x \cdot m)]$$

$\forall t, s \in S, \forall m \in M \quad \forall x \in R$

(Check that this is well-defined and the axioms are satisfied.)

2. $\forall$ short exact sequence

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

of R-modules,

$$0 \to S^{-1}M' \xrightarrow{\alpha_1} S^{-1}M \xrightarrow{\beta_1} S^{-1}M'' \to 0$$

is a short exact sequence of $S^{-1}R$-modules.

Suppose $\quad s^{-1}m \in M$, $\quad s^{-1}m := [(s, m)]$

and $\quad \beta_1(s^{-1}m) = 0$ in M''

$\beta_1(s^{-1}m) = s^{-1} \cdot \beta_1(m) = 0$

$\iff \exists t \in S$ s.t. $t \cdot \beta_1(m) = 0$, $\quad t \cdot \beta_1(m) = \beta_1(tm)$

$\Longrightarrow \exists\, m' \in M'$ s.t. $tm = \alpha(m')$

$$\Longrightarrow \quad s^{-1}m = \alpha_1(s^{-1}t^{-1}\cdot m') = \alpha_1([(st, m')])$$

3. $\forall$ R-module M, have $\qquad R \to S^{-1}R$

- an $S^{-1}R$-module $S^{-1}M$
- another $S^{-1}R$-module, $(S^{-1}R) \otimes_R M$
- an R-module homom

$$\begin{array}{ccc} M & \longrightarrow & S^{-1}M \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ m & \longmapsto & [(1, m)] \end{array} \qquad \forall m \in M$$

which gives rise to a $S^{-1}R$-module homom such that

$$\underline{\mathrm{can}_{S,R,M}}: S^{-1}R \otimes_R M \longrightarrow S^{-1}M$$

$$(s^{-1}x) \otimes m \longmapsto s^{-1}\cdot xm \qquad \text{(exer.)}$$

$$\begin{array}{l} \forall s \in S \\ \forall x \in R \\ \forall m \in M \end{array}$$

__Proposition__. $\mathrm{can}_{S,R,M}$ is an __isomorphism__!

__Pf__: Clear if M is a free R-module.

$$\because \qquad S^{-1}R \otimes_R R = S^{-1}R$$

and $M \rightsquigarrow S^{-1}R \otimes_R M$ both commute with

$M \rightsquigarrow S^{-1}M$ arbitrary direct sums.

Take a R-module epimorphism f_1

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}(f_1) \arrow[r] & F_1 \arrow[r, "f_1"] & M \arrow[r] & 0 \\
 & F_2 \arrow[u, "\bar{f}_2"] \arrow[ur, "f_2"'] & & &
\end{tikzcd}$$

and an R-module epimorphism $\bar{f}_2 : F_2 \to \operatorname{Ker}(f_1)$ with F_1 and F_2 free

$$\rightsquigarrow \quad F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} M \to 0 \quad \text{exact}$$

$\rightsquigarrow$ exact (as tensor product is right exact)

$$\begin{tikzcd}
S^{-1}R \otimes_R F_2 \arrow[r] \arrow[d, "\cong"] & S^{-1}R \otimes_R F_1 \arrow[r] \arrow[d, "\cong"] & S^{-1}R \otimes_R M \arrow[r] \arrow[d, "can"] & 0 \\
S^{-1}F_2 \arrow[r] & S^{-1}F_1 \arrow[r] & S^{-1}M \arrow[r] & 0
\end{tikzcd}$$

exact $\rightarrow$

$\Rightarrow$ can is an isomorphism! q.e.d.

To construct an arrow

$$\begin{tikzcd}
{[(1,m)]} \in S^{-1}M \arrow[r, dashed, "\chi"] & S^{-1}R \otimes_R M \\
M \arrow[u, "\xi"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

map $m \mapsto [(1,m)]$, $\quad m \mapsto 1 \otimes m \quad \forall m \in M$

check: ξ, η are R-module homom.

Moreover: every elt $s \in S$ operates on $S^{-1}R \otimes_R M$ as an R-module automorphism

$\rightsquigarrow$ η factors through ξ uniquely!

i.e. $\exists!\ \chi : S^{-1}M \longrightarrow S^{-1}R \otimes_R M$ $(S^{-1}R)$-linear s.t. $\eta = \chi \circ \xi$.

- $S^{-1}R \xleftarrow{\text{ring homom}} R$ is **flat**

$S^{-1}R$ is a flat R-algebra

$S^{-1}R$ is a flat R-module (via the ring homom $R \to S^{-1}R$)

$$\iff \underbrace{S^{-1}R \otimes_R ?}_{\text{is exact}} : M \mapsto S^{-1}R \otimes_R M$$

$\uparrow$ (R-modules)

$$\| \qquad S^{-1}(?) \qquad \underbrace{M \mapsto S^{-1}M}_{\text{is exact}}.$$

$$\mathbb{C}[X,Y] \ni \underline{X^2+Y^2}$$

$\|$ polynomial functions on $\mathbb{C}^2$.

invert $\underline{X^2+Y^2}$

$$\mathbb{C}[X,Y]\left[\frac{1}{X^2+Y^2}\right] \subseteq \mathbb{C}(X,Y) = \operatorname{frac} \mathbb{C}[X,Y]$$

$$= \left((X^2+Y^2)^N\right)^{-1} \mathbb{C}[X,Y]$$

exer $\diagup\!\!\diagup$

All rationals functions in $\mathbb{C}(X,Y)$ which are holomorphic in $\mathbb{C}^2 \setminus \left\{ \underset{\in \mathbb{C}^2}{(a,b)} \,\middle|\, a^2+b^2=0 \right\}$