Group rings

G: a group

$R[G]$ — free R-module with basis indexed by elements of G

(R commutative)

— typical element of $R[G]$:

$$\sum_{\text{finite}} a_i[x_i] \qquad a_i \in R,\ x_i \in G$$

$$\sum_i a_i[x_i] \cdot \sum_j b_j[y_j] = \sum_{i,j} a_i b_j [x_i \cdot y_j]$$

$$= \sum_{z \in G} \sum_{x_i \cdot y_j = z} a_i a_j [z]$$

Equivalent notions)

(1) a left module over the ring $R[G]$

(2) a group homom $\rho: G \longrightarrow \mathrm{End}_R(M)^\times$
where M is an R-module

(3) $\mu: G \times M \longrightarrow M$ $\qquad M$: an R-module

s.t. $\mu(1_G, m) = m \quad \forall m \in M$

$\mu(x, \mu(y, m)) = \mu(x \cdot y, m) \quad \forall x, y \in G\ \forall m \in M$

an "R-linear representation of G".

Example $\quad R = \mathbb{Q} \qquad G = \mathbb{Z}/5\mathbb{Z}$

Classify all irreducible $\mathbb{Q}$-representations of G?

$\longleftrightarrow$ simple $\mathbb{Q}[\mathbb{Z}/5\mathbb{Z}]$-modules.

$$\mathbb{Q}[\mathbb{Z}/5\mathbb{Z}] \cong \mathbb{Q}[x]/(x^5-1) \qquad \mathbb{Q}[\mathbb{Z}] \cong \mathbb{Q}[x, x^{-1}]$$

$$[n] \longleftrightarrow x^n$$

$$= \mathbb{Q}[x]/(x-1) \times \mathbb{Q}[x]/(x^4+x^3+x^2+x+1)$$

irreducible over $\mathbb{Q}$

2 simple modules:

- the trivial $(\mathbb{Z}/5\mathbb{Z})$-module
- a unique irreducible 4-dim$^{\ell}$ representation over $\mathbb{Q}$, corresponding to $\mathbb{Q}[x]/(x^4+\dots+1) \cong \mathbb{Q}(\zeta_5)$

$$G \geqslant H$$
$$\vee\!/$$
$$K \qquad \{gKg^{-1} \mid g \in G,\ gKg^{-1} \leqslant H\}$$

$GL_3(\mathbb{F}_2)$

Categories & functors

Category: $(\mathcal{C}, \mathrm{Mor})$

- a collection of "objects". $Ob(\mathcal{C}) = Ob_{\mathcal{C}}$
- a collection of "morphisms" $\mathrm{Mor}(\mathcal{C}) = \mathrm{Mor}_{\mathcal{C}}$

$$\mathrm{Mor}(\mathcal{C}) \overset{s}{\underset{t}{\rightrightarrows}} \mathrm{Ob}\,\mathcal{C}$$

$\Downarrow$

g, f

$s(g) = t(f) \rightsquigarrow g \circ f$

$\nearrow$ $\mathrm{Mor}(\mathcal{C})$

$$\mathrm{Mor}(\mathcal{C}) \underset{(s, \mathrm{Ob}(\mathcal{C}), t)}{\times} \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathcal{C})$$

- composition of morphisms : associative
- $1_X \in \mathrm{Mor}(X, X)$, $X \in \mathrm{Ob}$ $f \circ 1_X = f$, $1_X \circ g = g$

<u>small</u> category :
- $\mathrm{Mor}(X, Y)$ is a <u>set</u> $\forall X, Y \in \mathcal{C}$
- <u>sometimes</u> $\mathrm{Ob}(\mathcal{C})$ is a <u>set</u>

"<u>universe</u>"

<u>Example</u> : $\mathcal{C}$ = the category of all Riemann surfaces of genus 3

$\nearrow$ oriented, connected 2-dim'l manifolds with a conformal structure.

= smooth connected complete algebraic curves / $\mathbb{C}$ of genus 3.

Want to

Classify all genus 3 Riemann surfaces

families of over complex manifolds.

i.e.

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ S \end{array}$$

fibers are compact RS of genus 3

$S \leftarrow$ cpx mfds

Categories: (CpxMfd) = the category of all complex manifolds

$$(\text{Cpx Mfd}) \longrightarrow (\text{Sets})$$

$$S \longmapsto \frac{\{\text{the set of all families of genus 3 cpx manifolds}\}}{\{\text{isomorphism}\}}$$

Question is: Is this functor representable

i.e. ? $\exists$ a

$$\begin{array}{c} \mathcal{C}_{univ} \\ \downarrow \\ \mathcal{M}_{univ} \end{array}$$

$\leftarrow$ genus 3 RS

s.t. every family of genus 3 RS is

$$\cong \; g^*\begin{pmatrix} \mathcal{C}_{univ} \\ \downarrow \\ \mathcal{M}_{univ} \end{pmatrix} \quad \text{for a unique}$$

$$g : S \longrightarrow \mathcal{M}_{univ}$$

$$\in \mathrm{Mor}(S, \mathcal{M}_{univ})$$