

Group rings

G : a group

$R[G]$ - free R -module with basis indexed by elements of G
commutative
- typical element of $R[G]$:

$$\sum_{\text{finite}} a_i [x_i] \quad \begin{array}{l} a_i \in R \\ x_i \in G \end{array}$$

$$\begin{aligned} \sum_i a_i [x_i] \cdot \sum_j b_j [y_j] &= \sum_{i,j} a_i b_j [x_i \cdot y_j] \\ &= \sum_{z \in G} \sum_{x_i \cdot y_j = z} a_i b_j [z] \end{aligned}$$

Equivalent notions

(1) a left module over the ring $R[G]$

(2) a group homom $\rho: G \rightarrow \text{End}_R(M)^*$
where M is an R -module

(3) $\mu: G \times M \rightarrow M$ M : an R -module

s.t. $\mu(1_G, m) = m \quad \forall m \in M$

$$\mu(x, \mu(y, m)) = \mu(x \cdot y, m) \quad \begin{array}{l} \forall x, y \in G \\ \forall m \in M \end{array}$$

an " R -linear representation of G ".

Example $R = \mathbb{Q}$ $G = \mathbb{Z}/5\mathbb{Z}$

Classify all irreducible \mathbb{Q} -representations of G ?

\leftrightarrow simple $\mathbb{Q}[\mathbb{Z}/5\mathbb{Z}]$ -modules.

$$\mathbb{Q}[\mathbb{Z}/5\mathbb{Z}] \cong \mathbb{Q}[x]/(x^5-1)$$

$$\mathbb{Q}[\mathbb{Z}] \cong \mathbb{Q}[x, x^{-1}]$$

$$[n] \leftrightarrow x^n$$

$$= \mathbb{Q}[x]/(x-1) \times \mathbb{Q}[x]/(x^4+x^3+x^2+x+1)$$

2 simple modules:

\uparrow
irreducible over \mathbb{Q}

- the trivial $(\mathbb{Z}/5\mathbb{Z})$ -module

- a unique irreducible 4-dimensional representation over \mathbb{Q} ,
corresponding to

$$\mathbb{Q}[x]/(x^4+x^3+x^2+x+1) \cong \mathbb{Q}(\zeta_5)$$

$$G \supseteq H$$

$$\downarrow$$

$$K$$

$$\{gKg^{-1} \mid g \in G, gKg^{-1} \subseteq H\}$$

$$\underline{GL_3(\mathbb{F}_2)}$$

Categories & functors

Category: $(\mathcal{C}, \text{Mor})$

- a collection of "objects". $\text{Ob}(\mathcal{C}) = \text{Ob}_{\mathcal{C}}$

- a collection of "morphisms" $\text{Mor}(\mathcal{C}) = \text{Mor}_{\mathcal{C}}$

$$\text{Mor}(C) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \text{Ob} C$$

↓

g, f

$$s(g) = t(f)$$

↗

$g \circ f$

↑
Mor(C)

$$\text{Mor}(C) \times \text{Mor}(C) \longrightarrow \text{Mor}(C)$$

$(s, \text{Ob}(C), t)$

• composition of morphisms: associative

$$\begin{array}{l} \bullet 1_X \in \text{Mor}(X, X), \quad f \circ 1_X = f \\ X \in \text{Ob} \quad \quad \quad 1_X \circ g = g \end{array}$$

small category:

- $\text{Mor}(X, Y)$ is a set $\forall X, Y \in C$

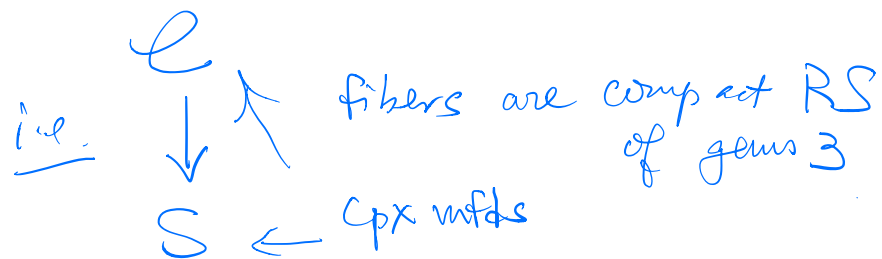
- sometimes $\text{Ob}(C)$ is a set

"universe"

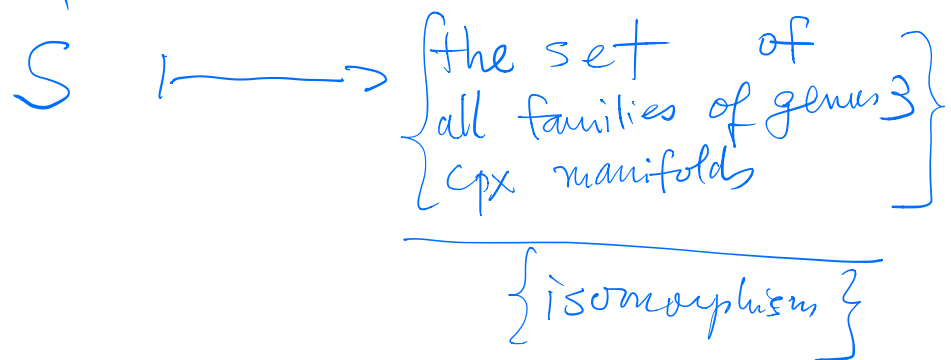
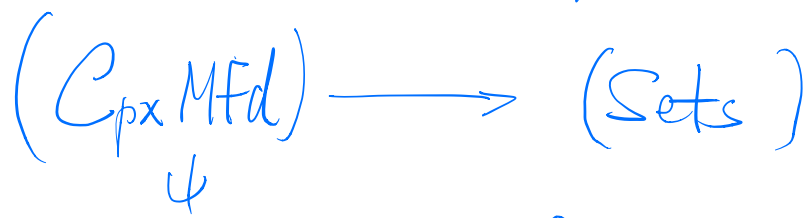
Example: $C =$ the category of all Riemann surfaces of genus 3

↑
oriented, connected
2-dim^l manifolds with a
conformal structure.

= smooth connected complete
 algebraic curves / \mathbb{C} of
 genus 3
 Want to Classify all genus 3 Riemann surfaces.
 Families of over complex manifolds.



Categories: $(\text{Cpx Mfd}) =$ the category of all complex manifolds



Question is: Is this functor representable



s.t. every family of gens \exists RS is

$$\cong g^* \left(\begin{array}{c} E_{\text{univ}} \\ \downarrow \\ M_{\text{univ}} \end{array} \right)$$

for a unique

$$g: S \rightarrow M_{\text{univ}} \\ \in \text{Mor}(S, M_{\text{univ}})$$