$\mathcal{C}$: additive category

$A \xrightarrow{u} B$ in $\mathcal{C}$

Ker(u), coker(u): defined by the usual universal properties

Assume: i) $\mathrm{Ker}(u) \xrightarrow{k} A \xrightarrow{u} B$

and $A \xrightarrow{u} B \xrightarrow{c} \mathrm{Coker}(u)$ exist

ii) $\mathrm{Ker}(u) \xrightarrow{k} A \longrightarrow \mathrm{coim}(u) \overset{def}{=} \mathrm{coker}(k)$

$\mathrm{Im}(u) \overset{def}{=} \mathrm{Ker}(c) \longrightarrow B \xrightarrow{c} \mathrm{Coker}(u)$ exist

⇝ Have

$$\begin{tikzcd}
& \mathrm{Ker}(u) \arrow[d, "k"] & \mathrm{Coker}(u) & \\
& A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[u, "c"] & \\
\mathrm{coker}(k) \arrow[r] & \mathrm{Coim}(u) \arrow[r, "\overset{?}{\cong}"'] & \mathrm{Im}(u) \arrow[u] & \mathrm{Ker}(c) \arrow[l]
\end{tikzcd}$$

Def: An abelian category $\mathcal{C}$ is an additive category such that

(0) finite coproducts exist

($\Rightarrow$ finite products exist. $\cong$ finite coproduct)

(1) Kernels and Cokernels of morphisms exist

(2) $\forall$ morphism $u \in \mathcal{C}$, the natural morphism

$$\mathrm{coim}(u) \longrightarrow \mathrm{im}(u)$$

is an isomorphism.

<u>Theorem</u> (Freyd)

Every abelian category $\mathcal{C}$ is isomorphic to an exact full subcategory of $\mathrm{Mod}(R)$ for some ring R

i.e. $\exists$ $\mathcal{C} \xrightarrow{j} \mathrm{Mod}(R)$

j: functor

fully faithful

<u>and</u> respects short exact sequences

$$0 \to X \to Y \to Z \to 0$$

is short exact in $\mathcal{C}$

iff

$$0 \to j(X) \longrightarrow j(Y) \to j(Z) \to 0$$

is short exact in $\mathrm{Mod}(R)$

$$\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D} \qquad \mathcal{F}: \text{ a functor}$$

$\mathcal{F}$ is faithful $\iff$ $\forall\, X, Y \in Ob(\mathcal{C})$ have
resp. fully faithful

$$Mor_{\mathcal{C}}(X,Y) \longrightarrow Mor_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

if this map is an injection (of sets)
resp. a bijection (of sets)

$\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is faithful
$\rightsquigarrow$ $\mathcal{F}$ embeds $\mathcal{C}$ as a subcategory

$\mathcal{F}$ is fully faithful $\rightsquigarrow$ $\mathcal{F}$ embeds $\mathcal{C}$ as a full subcategory

Many abelian category have some of the following properties:

- arbitry coproducts exist (products)
- have enough projectives
- have enough injectives

$\rightsquigarrow$ Define / construct derived functors.

<u>Suppose</u> $\mathcal{C}$: abelian category
with enough injectives

$T : \mathcal{C} \longrightarrow \mathcal{D}$ $\quad$ $\mathcal{D}$: abelian category
additive, left exact functor

$\rightsquigarrow$ Can define higher derived functors

$(R^i T)_{i \in \mathbb{N}}$

together with $T \xrightarrow{\sim} R^0 T$

such that

$\forall$ short exact sequence
$0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$,

have a long exact sequence

$$0 \to R^0T(X') \to R^0T(X) \to R^0T(X'')$$
$$\to R^1T(X') \to R^1T(X) \to R^1T(X'')$$
$$\to \cdots$$

<u>Construction</u>: Given $X \in \mathrm{Ob}(\mathcal{C})$

Pick an injective resolution
$0 \to X \to I^0 \to I^1 \to \cdots \to I^n \to I^{n+1} \to \cdots$
in $\mathcal{C}$.

define. $R^i T(X) \overset{\text{def}}{=} H^i\left(\left(T(I^\bullet)\right)_{\bullet\in\mathbb{Z}} \right)$

Similarly, if $S : \mathcal{C} \longrightarrow \mathcal{D}$ ← abelian

is a contravariant left exact functor

and $\mathcal{C}$ has enough projective.

⇝ Can define right derived functors

$$\left(R^i S\right)_{i\geq 0}$$

using projective resolutions in $\mathcal{C}$.

<u>Motivation</u>:

Historically: for application in the theory of <u>sheaves</u>.

<u>Def</u>^{n} of sheaves and presheaves:

<u>Def</u>^{n}: (presheaves on a topological space)

X: a topological space.

<u>Consider</u> $\mathrm{Top}(X) =$ "the category of open subsets of X".

i.e. Objects are open subsets $U \subset X$

Morphisms :
$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, phantom, "\cap" rotate=-90] & V \arrow[d, phantom, "\cap" rotate=-90] \\ X \arrow[r, equal] & X \end{tikzcd}$$

$$\mathrm{Mor}(U, V) = \begin{cases} U \hookrightarrow V & \text{if } U \subseteq V \\ \phi & \text{if } U \not\subseteq V \end{cases}$$

A presheaf is a contravariant functor

$$\mathcal{F} : Top(X) \longrightarrow \begin{array}{l} (\text{Sets}) \\ (\text{Abel. groups}) \\ (\text{Groups}) \\ (R\text{-modules}) \end{array}$$

i.e. Have $\mathcal{F}(U)$ $\forall$ open $U \subseteq X$

and $\mathcal{F}(U) \longleftarrow \mathcal{F}(V)$ $\forall$ pair of opens $U \hookrightarrow V$

Examples : X : topological space.

$U \rightsquigarrow C(U, \mathbb{R})$ = $\mathbb{R}$-valued continuous functions on U

open $\cap$ X

+ $U \subseteq V$ $C(V, \mathbb{R}) \xrightarrow[\text{restriction map}]{} C(U, \mathbb{R})$

Similarly $D \underset{\text{open}}{\subseteq} \mathbb{C}$

$$\mathrm{Top}(D) \ni U \rightsquigarrow \mathcal{O}(U) = \text{holomorphic functions on } U$$

<u>Def^n</u>: A presheaf $\mathcal{F} : \mathrm{Top}(X) \to (?)$ is a <u>sheaf</u> if the following "gluing cond" <u>holds</u>:

$\forall\, V \underset{\text{open}}{\subseteq} X$, $\forall$ open covering $(U_i)_{i \in I}$ of V

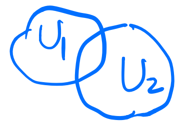
(i.e. U_i is open $\forall i$, $\bigcup_i U_i = V$)

$$\coprod_{(j,k) \in I \times I} U_j \cap U_k \rightrightarrows \coprod_{i \in I} U_i \longrightarrow V$$

$$0 \to \mathcal{F}(V) \to \prod_{i \in I} \mathcal{F}(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{(j,j') \in I \times I} \mathcal{F}(U_j \cap U_{j'})$$

is the inverse limit of α and β.

idea



case $I = \{1, 2\}$

$V = U_1 \cup U_2$

$$\mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1) \times \mathcal{F}(U_2) \rightrightarrows \mathcal{F}(U_1 \cap U_2)$$

$(s_1, \; s_2)$

if $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$

then s_1 and s_2 glue to an element in $\mathcal{F}(U_1 \cup U_2)$

"Tohoku paper" Math. J.

Grothendieck