Basic ideas of homological algebra of (left) modules

Elementary examples

$R = \mathbb{Z}$

0) $M = \mathbb{Z} \rightsquigarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, ?)$ is exact

$\rightsquigarrow \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, N) = 0$ for all $i \geq 1$ and all $\mathbb{Z}$-module N.

? $\mathrm{Ext}^i_{\mathbb{Z}}(?, \mathbb{Z}) = 0 \quad \forall_{i \geq 1}$?

free proj. resolution of $\mathbb{Z}/n\mathbb{Z}$ of length 1

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = ? \qquad \begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[u] \arrow[r, "{[n]}"] & \mathbb{Z} \arrow[u, "="'] \arrow[r] & 0 \end{tikzcd}$$

$0 \neq n \in \mathbb{Z}$

$\rightsquigarrow \quad \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0 \quad \forall_{i \geq 2}$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^i\Big(0 \to \underset{\substack{\| \\ \mathbb{Z} \\ i=0}}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})} \xrightarrow{[n]} \underset{\substack{\| \\ \mathbb{Z}}}{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})} \to 0\Big)$$

$$= \begin{cases} 0 & i=0 \\ \mathbb{Z}/n\mathbb{Z} & i=1 \end{cases}$$

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$$

$$\mathrm{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = H_i\Big(\big(0 \to \mathbb{Z} \xrightarrow{[n]} \mathbb{Z} \to 0\big) \otimes_{\mathbb{Z}} \mathbb{Z}\Big)$$

$$= H_i\big(0 \to \underset{\deg 1}{\mathbb{Z}} \xrightarrow{[n]} \underset{\deg = 0}{\mathbb{Z}} \to 0\big)$$

$\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$

(∵ $\mathbb{Z}$ is a projective $\mathbb{Z}$-module)

$$= \begin{cases} \mathbb{Z}/n\mathbb{Z} & i=0 \\ 0 & i=1 \end{cases}$$

Exer $\mathrm{Tor}_0^R(M_R, {}_RN) = M \otimes_R N$,

$\mathrm{Ext}^0_R({}_RM, {}_RN) = \mathrm{Hom}_R(M, N)$

$$\begin{aligned}
\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) &= H_i\Big(\big(0 \to \mathbb{Z} \xrightarrow{[n]} \mathbb{Z} \to 0\big) \otimes (\mathbb{Z}/n\mathbb{Z})\Big) \\
&= H_i\big(0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{[n]} \mathbb{Z}/n\mathbb{Z} \to 0\big) \\
&= \begin{cases} \mathbb{Z}/n\mathbb{Z} & i = 0, 1 \\ 0 & i \geq 2 \end{cases}
\end{aligned}$$

$\mathrm{Tor}_i^R(M_R, {}_RN)$
- use a proj. resol^n of M
- use a proj. resol^n of N
- use both

$P_\bullet \to M \to 0$ proj. resolution of M

$Q_\bullet \to N \to 0$ proj. resolution of N

$(\cdots \to P_2 \to P_1 \to P_0) =: P_\bullet$

$$H_i(P_\bullet \otimes_R N) = H_i(M \otimes_R N)$$

$$\simeq \qquad \simeq$$

$$= H_i\Big(\overbrace{s\big(\underbrace{P_\bullet \otimes_R Q_\bullet}_{\text{a double chain complex}}\big)}^{\text{the simplex chain complex associated to this double complex}}\Big)$$

__Def^n__: $C_{\bullet\bullet} = (C_{ij}, d_{ij}, \partial_{ij})$

$i,j \in \mathbb{N}$.

$$\begin{tikzcd} C_{i+1,j} \arrow[r, "d_{ij}"] & C_{i,j} \\ C_{i+1,j+1} \arrow[u, "\partial_{i+1,j}"] \arrow[r, "d_{i,j+1}"'] \arrow[ur, phantom, "-1"] & C_{i,j+1} \arrow[u, "\partial_{i,j}"'] \end{tikzcd}$$

a double chain complex

$$\begin{tikzcd} s(C_{\bullet\bullet})_k = \bigoplus_{i+j=k} C_{i,j} & dx + (-1)^a \partial x \\ s(C_{\bullet\bullet})_{k+1} = \bigoplus_{a+b=k+1} C_{a,b} \arrow[u] & x \in C_{a,b} \arrow[u, mapsto, "D"'] \end{tikzcd}$$

$D \circ D = 0$

ideas of

More general <u>homological algebra</u>

<u>Abelian categories</u>:

Have talked about categories:

1. Additive categories

A category $\mathcal{C}$ is an additive category if

(a) $\forall X, Y \in Ob(\mathcal{C})$,

$Mor_{\mathcal{C}}(X, Y)$ has a natural structure as an abelian group

(b) $\forall X, Y, Z \in Ob(\mathcal{C})$.

the map

$$\mathrm{Mor}_C(X,Y) \times \mathrm{Mor}_C(Y,Z) \to \mathrm{Mor}_C(X,Z)$$
$$(\alpha \quad , \quad \beta) \longmapsto \beta \circ \alpha$$

is $\mathbb{Z}$-bilinear.

Abelian category

C: additive $X, Y \in Ob(C)$, $\alpha \in \mathrm{Mor}_C(X,Y)$

$$\begin{tikzcd} X \arrow[r, "\alpha"] & Y \arrow[d, "c"] \\ K \arrow[u, "k"] & C \end{tikzcd}$$

Want: existence of $\mathrm{Ker}(\alpha)$: a sub-object of X

i.e. a mono $K \to X$

characterized by a suitable universal property.

Image

$\mathrm{Coker}(\alpha)$ = an epimorphism $Y \to C$ satisfying some suitable universal property

$\mathrm{Coker}(k)$ $\ker(c)$

+ identification of image with $\ker(\mathrm{coker})$. etc.