Math 603 Assignment 9, 2020-21

1. Consider the finite Hisenberg group H_p defined in assignment 8:

$$H_p := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \; : \; a, b, c \in \mathbb{F}_p \right\},$$

where p is a prime number. Let U be the subgroup of H_p consisting of all element $x \in H_p$ whose $(2,3)$-entry "c" vanishes. This subgroup U is commutative, which has p^2 one-dimensional characters. For each one-dimensional character χ of U, we have an induced character $\rho_\chi := \mathrm{Ind}_U^{H_p}(U, \chi)$.

(a) Determine explicitly the set of all one-dimensional characters χ of U such that ρ_χ is irreducible.

(b) For each χ such that ρ_χ is irreducible, determine all characters χ' of U such that $\rho_{\chi'} = \rho_\chi$.

(c) For each χ such that ρ_χ is reducible, find the decomposition of ρ_χ as a sum of irreducible characters of H_p.

2. The symmetric group S_4 contains a normal subgroup K isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and a subgroup S_3 consisting of all permutations of $\{1, 2, 3, 4\}$ fixing the letter 4; S_4 is a semi-direct product of K and S_3. In particular the quotient group S_4/K is isomorphic to S_3.

(a) Compute the character table of S_4. Note S_4 has 5 conjugacy classes; $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$, and S_4 has two 3-dimensional irreducible characters.

(b) Determine whether either of the two 3-dimensional irreducible characters χ_4 and χ_5 is induced from a one-dimensional character of a Sylow 2-subgroup P of S_4.

3. Let $D_n = (\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the dihedral group with $2n$ element, $n \geq 3$. Denote by N the normal subgroup $\mathbb{Z}/n\mathbb{Z}$. For each one-dimensional character χ of N, we have an induced character $\mathrm{Ind}_N^{D_n}(\chi)$ of degree 2.

(a) Determine explicitly which ones of the induced characters $\mathrm{Ind}_N^{D_n}(\chi)$ are irreducible.

(b) Determine when two induced characters $\mathrm{Ind}_N^{D_n}(\chi)$ and $\mathrm{Ind}_N^{D_n}(\chi')$ are equal.

(c) Compute the character table of D_n. (The shape of your answer will depend on the parity of n.)

4. Let H be a subgroup of a finite group G. The left action of G on G/H gives rize to a linear representation of G on $\mathbb{C}[G/H]$, often called the "permutation representation on G/H". Denote by $\tau_{G/H}$ the character of this representation.

(a) Show that $\sum_{x \in G} \tau_{G/H}(x) = \mathrm{card}(G)$.

(b) Show that $\sum_{x \in G} \tau_{G/H}(x)^2 = \mathrm{card}(H \backslash G/H) \cdot \mathrm{card}(G)$.

5. Recall that for a finite cyclic group $A \cong \mathbb{Z}/n\mathbb{Z}$, we defined $\mathbb{Z}$-valued functions θ_A and λ_A on A by

$$\theta_A(x) = \begin{cases} n & \text{if } x \text{ generates } A \\ 0 & \text{otherwise} \end{cases} \qquad \lambda_A := \phi(n)\mathrm{r}_A - \theta_A$$

where ϕ denotes Euler's ϕ-function, so $\phi(n) = \mathrm{card}(A^\times)$, and r_A is the character of the regular representation of A. We showed by direct computation that there exist positive integers $c_\psi \in \mathbb{N}_{>0}$ indexed by non-trivial one-dimensional characters ϕ of A such that

$$\lambda_A = \sum_\psi c_\psi \, \psi,$$

where ψ runs through all non-trivial one-dimensional characters A.

Prove the following explicit expression of the coefficients c_ψ:

$$c_\psi = \phi(n) - \frac{\phi(n)\,\mu(b)}{\phi(b)},$$

where $b = \mathrm{card}(\psi^{\mathbb{Z}})$ is the order of the character ψ, and μ is the Möbius function. Notice that $\phi(b)|\phi(n)$, and $\phi(b) = 1$ only when $b = 2$, in which case $c_\psi = 2\phi(n)$.

Recall that the Möbius function $\mu : \mathbb{Z}_{>0} \to \mathbb{N}$ is defined by specifying its generating function

$$\sum_{n \in \mathbb{N}_{>0}} \mu(n)\, n^{-s} = \prod_p (1 - p^{-s}),$$

where p runs through all prime numbers in the infinite product $\prod_p (1 - p^{-s})$.
(Hint: $\theta_A = \sum_\psi (\theta_A, \psi)\, \psi$, and the coefficient

$$(\theta_A, \psi) = \sum_{x \in A,\, x \text{ generates } A} \psi(x)$$

is an exponential sum. So this question asks for an explicit evaluation of this exponential sum.)

6. Let Q be the quaternion group. For each cyclic subgroup A of Q, we have a homomorphism

$$\mathrm{Ind}_A^G : R(A) \longrightarrow R(Q)$$

from the Grothendieck group $R(A)$ of virtual characters of A to the group $R(Q)$ of virtual characters of Q. Let $\mathcal{I}$ be the sum

$$\sum_{A \leq Q,\, A \text{ cyclic}} \mathrm{Ind}_A^Q(R(A)).$$

We showed in class that $[R(Q) : \mathcal{I}] < \infty$. Determine whether $\mathcal{I}$ is equal to $R(Q)$.