Math 602 Assignment 8, 2020-21

1. Compute the character table of the following finite groups:

(a) The quaternion group Q, the finite subgroup of the unit group $\mathbb{H}_{\mathbb{R}}^{\times}$ of the ring $\mathbb{H}_{\mathbb{R}}$ of all Hamilton's quaternions, defined by

$$Q := \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbb{H}_{\mathbb{R}}^{\times}.$$

(b) The dihedral group D_8 with 8 elements. Compare the character tables of D_8 and Q.

(c) The finite Heisenberg group $H_p \subseteq \mathrm{GL}_3(\mathbb{F}_p)$, defined by

$$H_p := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\},$$

where p is a prime number

2. Let n be a positive integer with $n \geq 3$. Define the generalize quaternion group Q_{2^n} with 2^n elements by

$$Q_{2^n} := \big((\mathbb{Z}/2^{n-1}\mathbb{Z}) \ltimes_{\rho_n} (\mathbb{Z}/4\mathbb{Z})\big)/\langle (2^{n-2} \bmod 2^{n-1}, 2 \bmod 2)\rangle,$$

where $\rho_n : \mathbb{Z}/4\mathbb{Z} \longrightarrow (\mathbb{Z}/2^{n-1}\mathbb{Z})^{\times}$ is the group homomorphism such that

$$\rho_n(b \bmod 4)(c \bmod 2^{n-1}) = (-1)^b \cdot c \bmod 2^{n-1} \quad \forall\, b, c \in \mathbb{Z},$$

and $\langle (2^{n-2} \bmod 2^{n-1}, 2 \bmod 2)\rangle$ is the subgroup of the semi-direct product

$$(\mathbb{Z}/2^{n-1}\mathbb{Z}) \ltimes_{\rho_n} (\mathbb{Z}/4\mathbb{Z})$$

generated by the element $(2^{n-2} \bmod 2^{n-1}, 2 \bmod 2)$ of order 2. Note that $\mathrm{card}(Q_{2^n} = 2^n$, and Q_8 is isomorphic to the quaternion group Q.

(a) Determine the character table of Q_{16}.

(b) (extra credit) Determine the character table of Q_{2^n} for $n \geq 5$.

3. Let p be a prime number and let $B(\mathbb{F}_p)$ be the subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$, consisting of all upper-triangular matrices in $\mathrm{GL}_2(\mathbb{F}_p)$. In particular $B(\mathbb{F}_p)$ has $p(p-1)^2$ elements.

(a) Show that $B(\mathbb{F}_p)$ has $p^2 - p$ conjugacy classes. Among them are p elements in the center, p conjugacy classes each with $p-1$ elements, and $(p-1)(p-2)$ conjugacy classes each with p elements.

(b) Show that $B(\mathbb{F}_p)$ has $(p-1)^2$ one-dimensional characters. So $B(\mathbb{F}_p)$ has $p-1$ non-abelian irreducible characters.

(c) Compute the character table of $B(\mathbb{F}_p)$ for $p = 3$.

(d) Compute the character table $B(\mathbb{F}_p)$ for general p's.

[Hint: If you want to "guess" the character table, there are many constraints which help. The orthogonality relations imply that every non-abelian irreducible character vanishes on those conjugacy classes with p-elements. Note also that the product of any irreducible character with a one-dimensional character is an irreducible character. So once you gave one irreducible non-abelian character, multiplication by one-dimensional characters give you $p-1$ characters, and you have gotten them all. You can also try to begin by constructing some non-trivial action of $B(\mathbb{F}_p)$ on some finite set. Alternatively try to construct an induced representation from a subgroup with relatively small index.]

4. (a) Compute the character table of $\mathrm{SL}_2(\mathbb{F}_3)$. Note that $\mathrm{card}(\mathrm{SL}_2(\mathbb{F}_p)) = p(p-1)(p+1)$.

(b) (extra credit) Compute the character table of $\mathrm{SL}_2(\mathbb{F}_5)$.

5. (extra credit) Compute the character table of the symmetric group S_5 and the alternating group A_5.

(Hint: The symmetric group S_4 is solvable and it is not difficult to compute the characters table of S_4. The group S_5 operate on the set $\{1,2,3,4,5\}$, giving rise to a 4-dimensional irreducible representation U of S_4; denote its character by χ_3. The product of the sign character χ_{sgn} with χ_3 is another irreducible character; call it χ_4. The second exterior product $\bigwedge^2 U$ is a 6-dimensional representation. Show that it is irreducible; call its character χ_7. Decompose the 10-dimensional second symmetric product $S^2(U)$ of U to get a 5-dimensional irreducible character χ_5. Multiply χ_5 with the sign character χ_{sgn} to get another 5-dimensional character χ_7. This gives the 7 irreducible characters of S_5. Decompose the restrictions to A_5 of irreducible representations of S_5 to get all irreducible characters of A_5.)