Math 602 Assignment 7, Fall 2020

Part A. From Gallier–Shatz:

- problem 121. Note: Subproblem 4 of this problem is a bit more difficult than the the first three, but also more interesting. It can be proved in at least three ways, one of which is quite elementary.

Part B.

1. (a) Let k be a field, and let G be a non-trivial finite subgroup of $k^\times$. Show that

$$\sum_{\gamma\in G}\gamma = 0$$

(an equality in k).

(b) Let k be a finite field, and let $f(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$ be a polynomial in n variables with coefficients in k. Find a formula for the sum

$$\sum_{(a_1,\dots,a_n)\in k^n} f(a_1,\dots,a_n)$$

in terms of coefficients of f.

2. Let n be a natural number with $n\geq 2$. Let $R=\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ be the integer group ring of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Determine the global projective dimension of R, i.e. $\sup\{\mathrm{proj.dim}_R(M)$, where M runs through all R-modules.

(You might want to start with the special cases $n=2$ and $n=3$, and some examples of R-modules.)

3. Let R be a ring (You may assume that R is commutative if you like). Let $(\Lambda,\leq)$ be a poset (partially ordered set) which is *directed*; c.f. 2.7 of Gallier–Shatz. Let $(M_\alpha,\phi_\alpha^\beta)$ be a inductive family of left R-modules indexed by Λ, and let $M=\varinjlim_{\alpha\in\Lambda}M_\alpha$. Let N be a left R-module. For each $i\in\mathbb{N}$, we have a projective system $\big(\mathrm{Ext}^i_R(M_\alpha,N)\big)_{\alpha\in\Lambda}$ indexed by Λ.

(a) Show that we have a natural map

$$\xi^i:\mathrm{Ext}^i_R(M,N)\longrightarrow\varprojlim_{\alpha\in\Lambda}\mathrm{Ext}^i_R(M_\alpha,N).$$

(b) Either prove that ξ^i is an isomorphism for all i, or give a counter-example.

4. (extra credit problem) Let R be a ring, and let n be a natural number. Suppose that the projective dimension $\mathrm{pd}_R(M)\leq n$ for every finitely generated left R-module M. (See Gallier–Shatz Definition 5.13 for the definition of projective dimension.) Does it follow that $\mathrm{pd}_R(M)\leq n$ for every left R-module N? Either give a proof, or give a counter-example.