Math 602 Assignment 6, Fall 2020

Part A. From Gallier–Shatz:

- problem 35
- problem 36
- problem 45
- problem 47
- problem 56
- problem 67
- problem 68
- problem 110 (Note: This problem has two parts. In the first part, the ring R is assumed to be a PID, and your answer should be a very explicit description/formula for the right derived functors of the functor t, where $t(M)$ is the R-submodule consisting of all torsion elements of M, for every R-module M. The second part depends on your answer for the first part.)

Part B.

1. (Extra credit problem) Let k be an algebraically closed field of positive characteristic p, where p is a prime number. Let $R = k[[x, y]]$, the power series ring over k in two variables x, y, with the topology such that $\{(xR+yR)^n\}_{n\in\mathbb{N}}$ is a fundamental system of open neighborhoods of 0 in R. Similarly let

$$S = R\hat{\otimes}_k R = k[[x_1, x_2, y_1, y_2]]$$

the completed tensor product of R with itself; the family $\{\mathfrak{m}_S^n\}_{n\in\mathbb{N}}$ of powers of the maximal ideal $\mathfrak{m}_S := x_1S + x_2S + y_1S + y_2S$ is a fundamental system of open neighborhoods of 0 in S. In the above

$$x_1 = x \otimes 1, \quad x_2 = 1 \otimes x, \quad y_1 = x_1 \otimes 1, \text{ and } y_2 = 1 \otimes y.$$

Let

$$\mu^* : R \to S$$

be the continuous k-linear ring homomorphism such that

$$\mu^*(x) = (1 + x) \otimes (1 + x) - 1 = x_1 + x_2 + x_1x_2,$$

$$\mu^*(y) = (1 + y) \otimes (1 + y) - 1 = y_1 + y_2 + y_1y_2.$$

Let $[p^2]^* : S \to S$ be the continuous k-linear ring endomorphism such that

$$[p^2]^*(x) = x^{p^2}, \quad [p^2]^*(y) = y^{p^2}.$$

Suppose that P is a prime ideal of R stable under $[p^2]^*$. Prove that

$$\mu^*(P) \subseteq (P \otimes 1) \cdot S + (1 \otimes P) \cdot S.$$

Remark. The complete local ring R is the coordinate ring of the standard two-dimensional formal torus $\mathbb{G}_m^2$ over k, $\mu^*R \to R\hat{\otimes}_k R$ is the group law of the $\mathbb{G}_m^2$, and $[p^2]^*$ correspond to the endomorphism "raising to the power p^2" on $\mathbb{G}_m^2$.