Math 602 Assignment 4, Fall 2020

Part A. From Gallier–Shatz:

- problem 37
- problem 39
- problem 42
- problem 48

Part B.

1. Let k be a field, and let K be an algebraic extension field of k. Let $\alpha, \beta, \gamma \in K$ such that $[k(\alpha) : k] = 2$ $[k(\beta) : k] = 3$ and $[k(\gamma) : k] = 7$.

(a) Find all possible values of $[k(\alpha+\beta) : k]$. (You need to produce examples for each asserted value, and prove that no other values are possible.)

(b) Find all possible values of $[k(\beta + \gamma) : k]$.

(Recall that $k(\alpha + \beta)$ denotes the smallest subfield of K which contains k and the element $\alpha + \beta$.)

2. Let $\mathbb{Z}[\sqrt{-5}]$ be the subring of $\mathbb{C}$ generated by $\mathbb{Z}$ and a square root of -5.

(a) Show that $\mathbb{Z}[\sqrt{-5}]$ is *not* a principal ideal domain.

(b) Give an example of a projective $\mathbb{Z}[\sqrt{-5}]$-module which is *not* free.

(c) Determine the projective dimension of $\mathbb{Z}[\sqrt{-5}]$, i.e. $\operatorname{suppr.dim}_{\mathbb{Z}[\sqrt{-5}]} M$, where M runs through all $\mathbb{Z}[\sqrt{-5}]$-modules.

3. Let k be a field, let K be a finitely generated extension field of k, and let F be a subfield of K. Is F necessarily finitely generated extension field of k? Either give a proof, or give a counter-example.

4. Let R be a commutative ring. Define the formal power series ring $R[[x]]$ in one variable x to be the set of all formal infinite series

$$\sum_{n \in \mathbb{N}} a_n\, x^n, \quad a_n \in R\ \forall\, n \in \mathbb{N}.$$

The binary operations "sum" and "product" in $R[[x]]$ are defined by the usual formulas. For instance

$$\Big(\sum_{n\in\mathbb{N}} a_n\, x^n\Big) \cdot \Big(\sum_{m\in\mathbb{N}} b_m\, x^m\Big) = \sum_{l\in\mathbb{N}} \Big(\sum_{0\le i\le l} a_i\, b_{l-i}\Big)\, x^l.$$

Assume that R is Noetherian. The Hilbert basis theorem asserts that the polynomial ring $R[x]$ is Noetherian; see Shatz–Gallier Theorem 2.11. Is it true that $R[[x]]$ is also Noetherian? Proof or counter-example.