# Math 602 Assignment 2, Fall 2020

**Part A.** From Gallier–Shatz: Problems 15, 18, 20, 21, 23, 25, 27.

Typo in problem 25: the symbol $\wr$ (for wreath products) in both parts of problem 25 should be replaced by $\rtimes$ (for semi-direct products).

**Part B.**

1. Let $A$ be an abelian group, not necessarily finite. Let $\mathbb{Z}[A]$ be the integer group ring attached to $A$. Recall that $\mathbb{Z}[A]$ is a free $\mathbb{Z}$-module with basis $\{[a] : a \in A\}$, and the product in $\mathbb{Z}[A]$ is determined by $[a] \cdot [b] = [a + b]$ for all $a, b \in A$.

(a) Show that if $A$ has a non-zero torsion element, then $\mathbb{Z}[A]$ is not an integral domain.

(b) Show that if $A$ is finitely generated, then $\mathbb{Z}[A]$ is an integral domain if and only if $A$ is torsion-free.

(c) Does the statement (b) hold without the assumption that $A$ is finitely generated? Either give a proof, or provide a counter-example.

2. Let $R$ be a commutative ring, and let $A = \sum_{i\in\mathbb{N}} A_i$ be a graded commutative $R$-algebra. (Our standing convention is that all rings have a unity element 1.)

(a) Suppose that $R = \mathbb{Z}$, $A_i$ is a finitely generated $\mathbb{Z}$-module for every $i \in \mathbb{N}$. Suppose moreover that the graded $\mathbb{F}_p$-algebra $A \otimes_{\mathbb{Z}} \mathbb{F}_p$ is isomorphic to a polynomial algebra $\mathbb{F}_p[x_a]_{a\in\mathbb{N}_{\geq 1}}$ in variables $x_a$, where $a$ ranges through all positive integers, and each $x_a$ is homogeneous of degree $a$. Prove that the graded algebra $A$ $A$ is isomorphic to the polynomial algebra $\mathbb{Z}[X_a]_{a\in\mathbb{N}_{\geq 1}}$ in variables $X_a$ homogeneous of degree $a$, $a \in \mathbb{N}_{\geq 1}$.

(b) Does the statement (a) hold without the assumption that each $A_i$ is a finitely generated $\mathbb{Z}$-module?

(c) Formulate and prove a statement for more general commutative base rings $R$ and and more general conditions than in (a).