Math 603 Assignment 13, 2020-21

Part I. The first 3 problems are meant to supplement the discussions in class related to change of groups. E.g. problem 2 provides an explicit quasi-isomorphism between $\mathrm{Res}^G_H(C_\bullet(G))$ and $C_\bullet(H)$. The actual questions in these three problems are straight-forward. Problems 4 and 5 gives some taste in dealing with concrete examples.

1. Let G be a group and let $H \leq G$ be a subgroup of G. Let N be a left H-module. We have two versions induced G-modules (from N):

$$\mathrm{ind}^G_H(N) := \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N,$$

and

$$\mathrm{Ind}^G_H(N) := \{f : G \to N \mid f(hx) = h \cdot f(x) \quad \forall\, h \in H, \forall\, x \in G,$$

where the left G-module structure on $\mathrm{Ind}^G_H(N)$ is given by

$$(y \cdot f)(x) := f(xy) \quad \forall\, f \in \mathrm{Ind}^G_H(N), \forall\, x, y \in G.$$

(a) For every element $g \in G$ and every element $n \in H$, define a function $f_{g,n} : G \to N$ supported on the coset $H \cdot g^{-1}$ by

$$f_{g,n}(x) = \begin{cases} (xg) \cdot n & \text{if } xg \in H \\ 0 & \text{if } xg \notin H. \end{cases}$$

Show that the "formula"

$$\sum_i [g_i] \otimes_H n_i \;\mapsto\; \sum_i f_{g_i,n_i}$$

gives a well-defined injective $\mathbb{Z}[G]$-linear map

$$\mathfrak{I}^G_H : \mathrm{ind}^G_H(N) \to \mathrm{Ind}^G_H(N).$$

(b) If $[G : H] < \infty$, then $\mathfrak{I}^G_H$ is an isomorphism, whose inverse is the map

$$\mathfrak{J}^G_H : \mathrm{Ind}^G_H(N) \to \mathrm{ind}^G_H(N)$$

which sends a typical element $f \in \mathrm{Ind}^G_H(N)$ to the element

$$\sum_{x \in H\backslash G} [x^{-1}] \otimes_H f(x) \in \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N.$$

(c) Either prove or disprove the following statements.

(c1) The map $H_i(G, \mathrm{ind}^G_H(N)) \to H_i(G, \mathrm{Ind}^G_H(N))$ induced by $\mathfrak{I}^G_H$ is an isomorphism for all H-module N and all $i \in \mathbb{N}$.

(c2) The map $H^i(G, \mathrm{ind}^G_H(N)) \to H^i(G, \mathrm{Ind}^G_H(N))$ induced by $\mathfrak{I}^G_H$ is an isomorphism for all H-module N and all $i \in \mathbb{N}$.

(Note that Shapiro's lemma gives canonical isomorphisms

$$H_i(G, \mathrm{ind}^G_H(N)) \xrightarrow{\sim} H_i(H, N)$$

and

$$H^i(G, \mathrm{Ind}^G_H(N)) \xrightarrow{\sim} H^i(G, N)$$

for every left H-module N and every $i \in \mathbb{N}$.)

(d) Let M be a left G-module, and let $\mathrm{Res}_H^G(M)$ be the left H-module with the same underlying abelian group as M and the H-action comes from the inclusion $H \hookrightarrow G$.

(d1) Show that there is a $\mathbb{Z}[G]$-linear surjective map

$$\theta_H^G : \mathrm{ind}_H^G(\mathrm{Res}_H^G(M)) \to M$$

such that

$$\theta_H^G : \sum_i u_i \otimes_H m_i \;\mapsto\; \sum_i u_i \cdot m_i$$

for all families $u_i \in \mathbb{Z}[G]$, $m_i \in M$ indexed by a finite set.
(The G-module homomorphism θ_H^G induces functorial maps

$$j(G \geq H, M) : H_n(H, \mathrm{Res}_H^G(M)) \xrightarrow{\simeq} H_n(G, \mathrm{ind}_H^G \mathrm{Res}_H^G M) \xrightarrow{\theta_{H*}^G} H_n(G, M)\ ,$$

called the *corestriction maps* for $H \leq G$ in group homology.)

(d2) Show that there is a $\mathbb{Z}[G]$-linear injective map

$$\psi_H^G : M \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$$

such that

$$\psi_H^G(m)(x) = x \cdot m \quad \forall m \in M \ \ \forall\, x \in G.$$

(The G-module map ψ_H^G induces functorial maps

$$i(H \leq G, M) : H^n(G, M) \xrightarrow{\psi_{H*}^G} H^n(G, \mathrm{Ind}_H^G \mathrm{Res}_H^G M) \xrightarrow{\simeq} H^n(H, \mathrm{Res}_H^G M)\ ,$$

called the *restriction maps* group group cohomology.)

2. Recall that for any group G, the inhomogeneous chain complex $(C_\bullet(G), \partial_\bullet)_{\bullet \in \mathbb{N}}$ is defined by

$$C_n(G) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G^n] \quad \forall\, n \in \mathbb{N},$$

with $\mathbb{Z}[G]$-module structure through the factor $\mathbb{Z}[G]$, so $C_n(G)$ is a free $\mathbb{Z}[G]$-module with basis

$$\big([\sigma_1, \ldots, \sigma_n] \,\big|\, \sigma_1, \ldots, \sigma_n \in G \big),$$

and also a free $\mathbb{Z}$-module with basis

$$\big(\sigma_0[\sigma_1, \ldots, \sigma_n] \,\big|\, \sigma_0, \sigma_1, \ldots, \sigma_n \in G \big).$$

The differential $\partial_n : C_n(G) \to C_{n-1}$, $n \geq 1$, is the $\mathbb{Z}[G]$-linear map defined by

$$\partial_n(\sigma_0[\sigma_1, \ldots, \sigma_n]) = \sigma_0\sigma_1[\sigma_2, \ldots, \sigma_n] + \sum_{i=1}^{n-1} (-1)^i \sigma_0[\sigma_1, \ldots, \sigma_{i-1}, \sigma_i\sigma_{i+1}] + (-1)^n \sigma_0[\sigma_1, \ldots, \sigma_{n-1}] \qquad \forall \sigma_0, \ldots, \sigma_n \in G.$$

The augmentation map $\epsilon : C_0(G) \to \mathbb{Z}$ is given by $\epsilon(\sigma_0) = 1\ \forall\, \sigma_0 \in G$; it defines a chain complex map from $((C_\bullet(G), \partial_\bullet)$ to the trivial chain complex $\mathbb{Z}$ concentrated at degree 0, which induces an isomorphism on homology groups. In other words $(C_\bullet(G), \partial_\bullet)$ is a free resolution of the trivial G-module $\mathbb{Z}$.

Let $H \leq G$ be a subgroup of G. The chain complex $(C_\bullet(G), \partial_\bullet)$, regarded as a complex of $\mathbb{Z}[H]$-modules, is another free resolution of the trivial H-module $\mathbb{Z}$, and one can use it to compute homology and cohomology groups of H-modules. Thus for every left H-module N, we have canonical isomorphisms

$$H_i(H, N) \cong H_i\big(N \otimes_H C_\bullet(G), \partial_\bullet\big), \quad \forall\, i \in \mathbb{N},$$

and similarly for cohomologies. Here $N \otimes_H C_\bullet(G)$ is short for $N \otimes_{\mathbb{Z}[H]} C_\bullet(G)$, and the tensor product $N \otimes_H C_\bullet(G)$ is formed by regarding N as a right H-module via the isomorphism $\tau \mapsto \tau^{-1}$ from H to H^{opp}, so that

$$\tau n \otimes_H \tau c \;=\; n \otimes_H c \;\in\; N \otimes_H C_\bullet(G) \quad \forall\, \tau \in H, \forall\, n \in H, \forall\, c \in C_\bullet(G).$$

(a) Let $\bar{s} : H\backslash G \to G$ be a section of the projection map $\pi : G \to H\backslash G$, i.e. $\pi \circ \bar{s} = \mathrm{id}_{H\backslash G}$. Let $s = \bar{s} \circ \pi$ be the composition of the projection π with the section $\bar{s}$. Define a map $\eta : G \to H$ (which depends on s) by

$$\eta(x) = x \cdot s(x)^{-1} \quad \forall\, x \in G,$$

i.e. $\eta(x) \cdot s(x) = x$ for all $x \in G$. Show that

$$s(xy) = s(s(x)y), \quad \eta(xy) = \eta(x)\eta(s(x)y) \quad \forall\, x, y \in G.$$

(b) Define maps $\phi_n : C_n(G) \to C_n(H)$, $n \in \mathbb{N}$, by

$$\begin{aligned}&\phi_n\big(\sigma_0\big[\sigma_1, \ldots, \sigma_n\big]\big) = \\ &\eta(\sigma_0)\big[\eta\big(s(\sigma_0)\sigma_1\big), \eta\big(s(\sigma_0\sigma_1)\sigma_2\big), \ldots, \eta\big(s(\sigma_0\sigma_1\cdots\sigma_{i-1})\sigma_i\big), \ldots, \eta\big(s(\sigma_0\cdots\sigma_{n-1})\sigma_n\big)\big].\end{aligned}$$

Show that the ϕ_n's define a morphism $(\phi_\bullet)$ in the category of chain complexes of left $\mathbb{Z}[H]$-modules, from $\big(C_\bullet(G), \partial_\bullet\big)$ to $\big(C_\bullet(H), \partial_\bullet\big)$, and is a quasi-isomorphism (i.e. $(\phi_\bullet)$ induces isomorphisms on all homology groups.)

[Note: This quasi-isomorphism $(\phi_\bullet)$ is convenient in explicit calculations.]

3. Suppose that $[G : H] < \infty$. Let M be a left G-module.

(a) Let

$$\Psi = \mathfrak{J}_H^G \circ \psi_H^G : M \to \mathrm{ind}_H^G(\mathrm{Res}_H^G(M))$$

be the composition of

$$\psi_H^G : M \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$$

with the isomorphism

$$\mathfrak{J}_H^G : \mathrm{Ind}_H^G \mathrm{Res}_H^G \xrightarrow{\sim} \mathrm{ind}_H^G \mathrm{Res}_H^G M.$$

Show that

$$\Psi(m) = \sum_{x \in H\backslash G} [x^{-1}] \otimes_{\mathbb{Z}[H]} x \cdot m$$

for all $m \in M$.

(b) Let

$$\Phi = \theta_H^G \circ \mathfrak{I}_H^G : \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)) \to M$$

be the composition of

$$\mathfrak{I}_H^G : \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M)) \xrightarrow{\sim} \mathrm{ind}_H^G(\mathrm{Res}_H^G(M))$$

with

$$\theta_H^G : \mathrm{ind}_H^G(\mathrm{Res}_H^G(M)) \to M.$$

Show that for each element $f : G \to M$ in $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$,

$$\Phi(f) = \sum_{x \in H\backslash G} x^{-1} \cdot f(x)$$

Definition. For each $n \in \mathbb{N}$, the *transfer map* (also called the restriction map) for group homology

$$\mathrm{Ver}_n^{H\leq G} : H_n(G, M) \to H_n(H, M)$$

is by definition the map which makes the following diagram

$$\begin{array}{ccccc} H_n(C_\bullet(G) \otimes_G M) & \xrightarrow{\Psi_*} & H_n\big(C_\bullet(G) \otimes_G \mathrm{ind}_H^G \mathrm{Res} M\big) & \xrightarrow{\simeq} & H_n(C_\bullet(G) \otimes_H M) \\ \uparrow{=} & & & & \simeq \downarrow \phi_{n*} \\ H_n(G, M) & \xrightarrow{\mathrm{Ver}_i^{H\leq G}} & H_n(H, M) & \xrightarrow{=} & H_n(C_\bullet(H) \otimes_H M) \end{array}$$

commutative.

Remark. (i) The transfer maps are functorial, i.e. it defines a morphism between δ-functors on the category left G-modules. So the transfer map $\mathrm{Ver}_0^{H\leq G}$ at degree 0 determines $\mathrm{Ver}_n^{H\leq G}$ for all $n \in \mathbb{N}$.

(ii) The tensor products $\otimes_G$ and $\otimes_H$ in the above diagram are formed with the general convention: when we take the tensor product A_1, A_2 of two left G-modules, we turn one of the factors into a right G-module by the isomorphism $G \xrightarrow{x \mapsto x^{-1}} G^{\mathrm{opp}}$, then take the tensor product, over $\mathbb{Z}[G]$, of a right $\mathbb{Z}[G]$-module with a left $\mathbb{Z}[G]$-module. So this tensor product $A_1 \otimes_G A_2$ is the same as the coinvariants $(A_1 \otimes_{\mathbb{Z}} A_2)_G$ for the diagonal left action of G on $A_1 \otimes_{\mathbb{Z}} A_2$. This reminder is relevant for part (c) below.

(c) Show that the formula

$$m \bmod I_G M \quad \mapsto \quad \sum_{y \in H\backslash G} y \cdot m \bmod I_H M,$$

gives a well-defined map

$$N_{H\backslash G} : M_G \to M_H.$$

Prove that the transfer map for the homology groups at degree 0

$$H_0(G, M) = M_G \xrightarrow{\mathrm{Ver}_0^{H\leq G}} M_H = H_0(H, \mathrm{Res}_H^G(M))$$

from M_G to M_H is equal to $N_{H\leq G}$.

(Hint: To see that the right-hand-side of above formula is well-defined, show that

$$N'_{H\backslash G} := \sum_{y\in H\backslash G} [y] \mod I_H \ \in \mathbb{Z}[G]/I_H\cdot\mathbb{Z}[G]$$

is a well-defined element in $\mathbb{Z}[G]/I_H\cdot\mathbb{Z}[G]$, and $N'_{H\backslash G}\cdot I_G \subseteq I_H\cdot\mathbb{Z}[G]$.)

Remark. (iii) As remarked at the end of part (b) above, one can also use this formula to *define* the transfer map on group homologies. The purpose of statement (c) is to verify that the two definitions give the same map.

(iv) The composition

$$H_n(G,M)\xrightarrow{\mathrm{Ver}_n^{H\leq G}} H_n(H,\mathrm{Res}^G_H M)\xrightarrow{j(G\geq H,M)} H_n(G,M)$$

is equal to $[[G:H]\cdot \mathrm{id}_{H_n(G,M)}$. This assertion is an immediate consequence of (c) when $n=0$. The general case follows from the case $n=0$, as in remark (ii), by degree shifting.

(d) In the case when $i=1$ and M is trivial G-module $\mathbb{Z}$, we have canonical isomorphisms $H_1(G,\mathbb{Z})\cong G^{\mathrm{ab}}$ and $H_1(H,\mathbb{Z})\cong H^{\mathrm{ab}}$. Let $\bar{s}: H\backslash G\to G$ be a section of the canonical projection $\pi: G\to H\backslash G$ as in problem 2 above, and let $s=\bar{s}\circ\pi$ and $\eta: G\to H$ be as in problem 2. Prove that

$$\mathrm{Ver}_1 : G^{\mathrm{ab}}\to H^{\mathrm{ab}}$$

is given by

$$\mathrm{Ver}_1(x \bmod (G,G)) = \prod_{\bar{y}\in H\backslash G} \eta(\bar{s}(\bar{y})x)) \bmod (H,H)).$$

(The product $\prod_{\bar{y}\in H\backslash G}$ modulo the commutator subgroup (H,H) is well-defined.)

(e) (extra credit) Formulate a similar description of the transfer maps (also called the corestriction maps) in group cohomology

$$\mathrm{Ver}^i_{H\leq G} : H^i(H,M)\to H^i(G,M)$$

for cohomologies, and find an explicit formula for

$$\mathrm{Ver}^1_{H\leq G} : H^1(H,\mathbb{Q}/\mathbb{Z})\to H^1(G,\mathbb{Q}/\mathbb{Z}).$$

(Hint: The cohomological analog of (c) uses the element

$$N_{G/H} := \sum_{x\in G/H} [x] \mod \mathbb{Z}[G]\cdot I_H$$

of $\mathbb{Z}[G]/\mathbb{Z}[G]\cdot I_H$.)

4. (a) Compute the cohomology groups $H^i(S_3,\mathbb{Z})$ (as abstract abelian groups) for $i=0,1,2$.

(b) (extra credit) Compute $\hat{H}^i(S_3,\mathbb{Z})$ for some $i\notin\{-2,-1,0,1,2\}$.

5. (a) Let $\mathrm{sgn}: S_3\to \mu_2=\mathbb{Z}^\times$ be the sign character of S_3. Compute the cohomology groups $H^i(S_3,\mathbb{Z}(\mathrm{sgn}))$ for $i=0,1,2$. Here $\mathbb{Z}(\mathrm{sgn}))$ is the S_3-module with $\mathbb{Z}$ as the underlying abelian group, such that S_3 operates via the sign character of S_3.

(b) (extra credit) Compute $\hat{H}^i(S_3,\mathbb{Z}(\mathrm{sgn}))$ for some $i\notin\{-2,-1,0,1,2\}$.

Part II. From Gallier–Shatz

- problem 134 (1), (2); part (3) is extra credit. (This problem is not directly related to cohomologies of groups.)
- problem 137. (This problem is about finding a *direct and explicit* description of the bijection between the set of all classes of extensions of G by M and the set of all classes of 2-extensions of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ and M, *not* using cocycles.)
- (extra credit) problem 140. (This problem is about identifying the cohomology group $H^i(G, M)$ with a Hochschild homology group, of a suitable module over $R \otimes R^{\mathrm{opp}}$, where $R = \mathbb{Z}[G]$.)