Math 603 Assignment 10, 2020-21

1. In the proof of Brauer's theorem on 2/08/2021, we use the following fact, proved by direct computation:

> *Let G be a finite group, and let $g := \mathrm{card}(G)$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{Q}(\mu_g)} = \mathbb{Z}[\mu_g]$. Suppose that f is a $\mathbb{Z}$-valued class function on a finite group G such that $f(x) \equiv 0 \pmod{g}$ for every $x \in G$. Then f is an $\mathcal{O}$-linear combination of characters of G induced from cyclic subgroups.*

If we change "$\mathcal{O}$-linear combination" to "$\mathbb{Z}$-linear combination" in the above statement, is the resulting statement true? (Either prove the statement is true, or prove that it is false.)

2. Let G be a finite group, and let $g := \mathrm{card}(G)$. Let p be a prime number. Let $\mathcal{O} = \mathcal{O}_{\mathbb{Q}(\mu_g)} = \mathbb{Z}[\mu_g]$. Let x be an element of G, and let $x = x_r\, x_s$ be the canonical decomposition of x as the product of its p-regular part and p-singular part, i.e. $x_r, x_s \in x^{\mathbb{Z}}$, the order of x is prime to p, and the order of x_s is a power of p.

In the proof of Brauer's theorem on 2/08/2021, we used the following statement. (The proof is based on the observation that $h(x)^{p^N} \cong h(x_r)^{p^N} \pmod{p\mathcal{O}}$, for any positive integer N such that $x_s^{p^N} = 1$.)

> *Let $h : G \to \mathbb{Z}$ be a $\mathbb{Z}$-valued class function on G which is an $\mathcal{O}$-linear combination of characters of G. Then $h(x) \equiv h(x_r) \pmod{p}$.*

If we delete the clause "which is an $\mathcal{O}$-linear combination of characters of G" from the above statement, is the resulting statement true? (Either prove the statement is true, or prove that it is false.)

3. Let G be a finite p-group. Let χ be an irreducible character of G. Show that the sum

$$\sum_{\psi \text{ irred}, \psi(1)<\chi(1)} \psi(1)^2 \equiv 0 \pmod{\chi(1)^2},$$

where ψ in the sum runs through all irreducible characters of G such that $\psi(1) < \chi(1)$.
(Hint: Recall that the degree of every irreducible character of a finite group G divides the cardinality of G.)

4. (a) Is the alternating group A_4 solvable? Is it supersolvable? Is it nilpotent?

(b) The group A_4 has an irreducible character χ with $\chi(1) = 3$. Determine whether χ is induced from a proper subgroup of A_4.

5. Let (V, std) be the standard 5-dimensional permutation representation of the symmetric group S_5. Splitting off a copy of the trivial representation of S_5, we get a 4-dimensional representation of U of S_5.

(a) Determine whether U is an irreducible representation of S_5.

(b) Determine whether the second exterior product $\bigwedge^2 U$ is in irreducible representation of S_5.

(c) Find all one-dimensional characters of S_5.

(Note that the product of an irreducible character of a finite group G with a one-dimensional character of G is again irreducible.)

(d) Let (W, η) be the permutation representation of S_5 corresponding to the action of S_5 on the set T consisting of all unordered pairs $\{a, b\}$, with $a \neq b \in \{1, 2, 3, 4, 5\}$. Show that (W, η) is isomorphic to the second symmetric product S^2U of the representation U of S_5.

(d) Determine whether (W, η) is an irreducible representation of S_5. If not, decompose the character of (W, η) as a sum of irreducible characters of S_5.

(f) Determine the character table of S_5 using results you found in (a)–(e). (You should find 7 irreducible characters; $1^2 + 1^2 + 4^2 + 4^2 + 5^2 + 5^2 + 6^2 = 120$.)

6. We use the notation in question 5 above.

(a) Determine whether the restriction $\mathrm{Res}^{S_5}_{A_5}(U)$ to the alternating subgroup $A_5 \leq S_5$ is an irreducible representation of A_5.

(b) Determine whether $\mathrm{Res}^{S_5}_{A_5}(\bigwedge^2 U)$ is an irreducible representation of A_5.

(c) For each irreducible character χ of S_5 with $\chi(1) \geq 2$, determine whether $\mathrm{Res}^{S_5}_{A_5}(\chi)$ is irreducible. In case $\mathrm{Res}^{S_5}_{A_5}(\chi)$ is, decompose $\mathrm{Res}^{S_5}_{A_5}(\chi)$ as a sum of irreducible characters of A_5.

7. We use the notation in question 5 above.

(a) Determine whether the representation U of S_5 is induced from a representation of a proper subgroup of S_5.

(b) Determine whether the representation $\mathrm{Res}^{S_5}_{A_5}(U)$ of A_5 is induced from a representation of a proper subgroup of A_5

(c) (extra credit) For each irreducible character χ of S_5 with $\chi(1) \geq 2$, determine whether χ is induced from a proper subgroup of S_5.

8. Let G be a finite group. For each cyclic subgroup A of G, let

$$\lambda_A := \phi(a)\,\mathrm{reg}_A - \theta_A,$$

where $\phi(a) = \mathrm{card}\big((\mathbb{Z}/a\mathbb{Z})^\times\big)$, reg_A is the regular representation of A, and θ_A is the $\mathbb{Z}$-valued function on A such that for any element $x \in A$, $\theta_A(x) = a$ if x generates A, and $\theta_A(x) = 0$ otherwise. Prove that

$$\sum_{A \leq G,\, A\,\mathrm{cyclic}} \mathrm{Ind}^G_A(\lambda_A) = \mathrm{card}(G) \cdot \big(\mathrm{reg}_G - 1\big).$$