Math 602 Assignment 1, Fall 2020

Part A. From Gallier–Shatz: Problems 1, 2, 6, 9.

Part B.

Definition Suppose a group G operates transitively on the left of a set X.

(i) We say that the action (G, X) is *imprimitive* if there exists a non-trivial partition Q of X which is stable under G. Here "non-trivial" means that

$$Q \neq \{X\} \quad \text{and} \quad Q \neq \{\{x\} : x \in X\}.$$

Such a partition Q will be called a *system of imprimitivity* for (G, X).

(ii) We say that the action (G, X) is *primitive* if it is not imprimitive.

1. (a) Suppose that (G, X, Q) is a system of imprimitivity for a transitive left action (G, X), $Y \in Q$, $y \in Y$. Let $H = \mathrm{Stab}_G(Y)$, $K = \mathrm{Stab}_G(y)$. Prove the following statements.

- $K < H < G$ and H operates transitively on Y.
- $|X| = |Y| \cdot |Q|$, $|Q| = [G : H]$, $|Y| = [H : K]$. (These statements hold even if X is infinite.)

(b) Suppose that (G, X) is a transitive left action, $y \in X$, $K := \mathrm{Stab}_G(y)$, and H is a subgroup of G such that $K < H < G$. Let $Y := H \cdot y \subset X$, and let $Q := \{g \cdot Y \mid g \in G\}$. (Our general notation scheme is that $\mathrm{Stab}_G(Y) := \{g \in G \mid g \cdot Y = Y\}$.) Show that (G, X, Q) is a system of imprimitivity.

(c) Suppose that (G, X) is a transitive left action, $x \in X$. Show that (G, X) is primitive if and only if G_x is a maximal proper subgroup of G. (Note that a maximal proper subgroup of G is often called a maximal subgroup of G.)

2. Let p be a prime number. For $n = 2, 3$, find a p-Sylow subgroup P of $\mathrm{GL}_n(\mathbb{F}_p)$. Also, determine the cardinality of $\mathrm{GL}_n(\mathbb{F}_p)$ and the normalizer of P in $\mathrm{GL}_n(\mathbb{F}_p)$.

3. Let (G, X) be a transitive left action, $x \in X$, $H := \mathrm{Stab}_G(x)$. Let $K \leq H$ be a subgroup of H. Let $\mathrm{Fix}(K) := \{x \in X \mid k \cdot x = x \; \forall \, k \in K\}$ be the fixed point subset of K in X. Let ${}^H K$ be the family of H-conjugates of K. Denote by ${}^G K \cap H$ the family of G-conjugates of K which lie in H. Show that $\mathrm{N}_G(K)$ operates transitively on $\mathrm{Fix}(K)$ if and only if ${}^H K = {}^G K \cap H$.

4. (a) Suppose that (G, X) is a transitive left G action, $x \in X$, and H is a subgroup of G. Let $K = \mathrm{Stab}_G(x)$. Show that H operates transitively on X if and only if $G = H \cdot K$. Here

$$H \cdot K := \{h \cdot k \mid h \in H, k \in K\}$$

(b) (Frattini argument) Let H be a subgroup of K, and let X be a subset of H. Denote by ${}^K X$ (resp. ${}^H X$) the set of all K-conjugates of X (resp. the set of all K-conjugates of X). Prove that $K = H \cdot \mathrm{N}_K(X)$ if and only if ${}^K X = {}^H X$.

(c) (Frattini argument) Let H be a normal subgroup of G, p be a prime number, and let P be a p-Sylow subgroup of H. Prove that $G = H \cdot \mathrm{N}_G(P)$.

(d) Let G be a finite group and let $\Phi(G)$ be the Frattini subgroup of G. Show that every Sylow subgroup of $\Phi(G)$ is normal in $\Phi(G)$. Conclude that $\Phi(G)$ is the direct product of its Sylow subgroups. Moreover $\Phi(G)$ has property N. (See §1.3 of Chap. 1 of Shatz–Gallier.)

5. Let n be a positive integer. Recall that a square matrix $A \in \mathrm{M}_n(\mathbb{C})$ is said to be *normal* if $A \cdot {}^t\bar{A} = {}^t\bar{A} \cdot A$, where ${}^t\bar{A}$ is the Hermitian conjugate of A. Determine the dimension of the subset $\mathcal{N}_n \subseteq \mathrm{M}_n(\mathbb{C})$ of normal matrices in $\mathrm{M}_n(\mathbb{C})$. (Your answer should include a reasonable definition of the notion of "dimension" of a subset of $\mathrm{M}_n(\mathbb{C})$, at least for the subset $\mathcal{N}_n$.)