Math 4250 Problem Set 9, Spring 2024

Part 1. From Strauss, *Partial Differential Equations.*

- Exercise 9.2, #12, page 241.
- Exercise 10.1, #2, #3 page 264

Part 2. Read pages 266–269 of Strauss, on Bessel functions. The definition of Bessel functions J_n of integral order is given on page 267 of Strauss.

2.1 Show that $\frac{d}{dx}J_0(x) = -J_1(x)$.

2.2 Show that

$$\frac{d}{dx}\big(x^n J_n(x)\big) = x^n J_{n-1}(x)$$

for all integers $n \geq 1$.

2.3 (extra credit) Show that

$$e^{x(t-t^{-1})} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)[t^n + (-1)^n t^{-n}] \qquad \forall t \neq 0.$$

(The function $e^{x(t-t^{-1})}$ is called the generating function of Bessel functions of integer order. When expanded in powers of t, the coefficients of t^n and t^{-n} are $J_n(x)$ and $(-1)^n J_n(x)$ for all $n \geq 0$. You can think of it as an easy way to "remember" the power series expansion of the J_n's.)

Part 3.

1. Let $u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)$ be smooth functions on $\mathbb{R}^3 \times \mathbb{R}$ which satisfy the wave equation. Here $\mathbf{x} = (x_1, x_2, x_3)$. Let $\phi_i(\mathbf{x}) = u_i(\mathbf{x}, 0)$, $\psi_i(\mathbf{x}) = \frac{\partial u}{\partial t}(\mathbf{x}, 0)$ for $i = 1, 2$. Suppose ϕ_i, ψ_i are functions of $\mathbf{x}$, i.e. there exists function f_i, g_i in one variable such that $\phi_i(\mathbf{x}) = f_i(|x|)$, $\psi_i(\mathbf{x}) = g_i(|x|)$, $i = 1, 2$. Let $r_0 > 0$ be a positive number such that $f_1(r_0) = f_2(r_0)$ and $g_1(r_0) = g_2(r_0)$.

(a) Is it necessarily true that $u_1(\mathbf{0}, r_0/c) = u_2(\mathbf{0}, r_0/c)$?

(b) (extra credit) Either give a complete proof that the answer to (a) is affirmative, or give a counter-example to show that the answer to (a) is negative.

2. (extra credit) It is known that the surface area (in dimension $n-1$) of the unit sphere in $\mathbb{R}^n$ is $2\pi^{n/2}/\Gamma(\frac{n}{2}) := A_n$, and the volume of the unit ball in $\mathbb{R}^n$ is $A_n/n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$. Here Γ is the Gamma function; see A.5 of Strauss.

Denote by $\mathbf{x} = (x_1, \ldots, x_n)$ the coordinate functions on $\mathbb{R}^n$. For any smooth function $u(\mathbf{x}, t)$ on $\mathbb{R}^n \times \mathbb{R}$, define the spherical mean function on $\mathbb{R}^4 \times \mathbb{R}_{>0} \times \mathbb{R}$ attached to u, by

$$I[u](\mathbf{x}, r, t) = \frac{1}{C_n r^{n-1}} \int_{|\mathbf{w}|=r} u(\mathbf{x}+\mathbf{w}, t)\, dS_{\mathbf{w}} = \frac{1}{C_n} \int_{|\mathbf{y}=1} u(\mathbf{x}+r\mathbf{y})\, dS_{\mathbf{y}},$$

where $dS_{\mathbf{w}}$ denotes the surface area element for the sphere $\{\mathbf{w} \in \mathbb{R}^n : |\mathbf{w}| = r\}$ of radius r in $\mathbb{R}^n$, and similarly for $dS_{\mathbf{y}}$.

(a) Show that

$$\frac{\partial}{\partial r} I[u](\mathbf{x}, r, t) = \frac{1}{r^{n-1}} \int_0^r I[\Delta u](\mathbf{x}, \rho, t) \rho^{n-1} d\rho.$$

(differentiate under the integral sign, then use the divergence theorem)

(b) Show that

$$\frac{\partial}{\partial r}\big(r^{n-1} \frac{\partial}{\partial r} I[u]\big)(\mathbf{x}, r, t) = r^{n-1} I[\Delta u](\mathbf{x}, r, t).$$

In other words,

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\Big) I[u] = I[\Delta u].$$

Here $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian on $\mathbb{R}^n$.

3. (extra credit, continue with the convention in problem 2 above) Suppose that $u(\mathbf{x}, t)$ satisfies the wave equation $\big(\frac{\partial^2}{\partial t^2} - c^2\Delta\big)u = 0$. Show that $I[u](\mathbf{x}, r, t)$ satisfies the differential equation

$$\Big(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2} - c^2\,\frac{n-1}{r}\frac{\partial}{\partial r}\Big) I[u](\mathbf{x}, r, t) = 0.$$

4. (extra credit) Suppose that n is an odd number, and let $n = 2k+1$, where k is an odd positive integer. Suppose that $u(\mathbf{x}, t)$ in problem 2 satisfies the wave equation $\big(\frac{\partial^2}{\partial t^2} - c^2\,\Delta\big)u = 0$. Show that the function

$$U(\mathbf{x}, r, t) = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \big(r^{2k-1} I[u](\mathbf{x}, r, t)\big)$$

on $\mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{R}$ satisfies the 1-dimensional wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2}\Big) U(\mathbf{x}, r, t) = 0.$$