Math 4250 Problem Set 8, Spring 2024

Part 1. From Strauss, *Partial Differential Equations.*

- Exercise 9.1, #1, #2, #3, page 233
- Exercise 9.2, #5, #11, pages 240–241

Part 2.

1. Let A be a symmetric positive definite $n \times n$ real matrix, i.e. $(x|Ax) > 0$ for all nonzero elements $x \in \mathbb{R}^n$. Consider the quadratic polynomial

$$Q(x) := \tfrac{1}{2}(x|Ax) - (b|x).$$

(a) Show that Q is bounded below, that is, there is a constant m so that $Q(x) \geq m$ for all $x \in \mathbb{R}^n$.

(b) Suppose that $x_0 \in \mathbb{R}^n$ minimizes Q; i.e. $(x|Ax) \geq (x_0|Ax_0)$. Show that $Ax_0 = b$.

(One approach: For any $y \in \mathbb{R}^n$, consider the function $Q(x_0 + \epsilon y)$ in ϵ, which has a minimum at $\epsilon = 0$. So the derivative with respect to ϵ vanishes at $\epsilon = 0$.)

[Moral: One way to solve $Ax = b$ is to minimize Q.]

2. Let $\Omega \in \mathbb{R}^3$ be a bounded region with smooth boundary and let $F(x)$ be a bounded continuous function on Ω. Also, let $\mathcal{S}$ be the set of smooth functions $u(x)$ on Ω that are zero on the boundary, i.e. $u(x) = 0$ for all $x \in \partial\Omega$. Define a function J on $\mathcal{S}$ by

$$J(u) := \int_\Omega \left[\tfrac{1}{2}|\nabla u|^2 + F(x)u\right] d^3x.$$

(Such a function J is sometimes called a "functional", a scaler-valued function on a space of functions.) Suppose that $u_0(x) \in \mathcal{S}$ minimizes J, i.e. $J(u) \geq J(u_0)$ for all $u \in \mathcal{S}$. Show that $\Delta u_0 = F$ in Ω – and of course $u_0 = 0$ on $\partial\Omega$.

(Hint: For any smooth function v on an open subset of $\mathbb{R}^3$ which contains the closure of Ω such that v vanishes on the boundary of Ω. For any real number ϵ, we have

$$\int_\Omega \left[\tfrac{1}{2}|\nabla u + \epsilon\nabla v|^2 + F(x)(u + \epsilon v)\right] d^3x \geq \int_\Omega \left[\tfrac{1}{2}|\nabla u|^2 d^3x + F(x)u\right] d^3x$$

Therefore the derivative with respect to ϵ of $\int_\Omega \left[\tfrac{1}{2}|\nabla u + \epsilon\nabla v|^2 + F(x)(u + \epsilon v)\right] d^3x$ vanishes. Integrate by parts, to get an integral of the form

$$\int_\Omega (\text{some function depending only on } u \text{ and } F) \cdot v(x)\, d^3x = 0,$$

for every function v satisfying the above condition.)

[Moral: One way to solve $\Delta u = F$ with $u = 0$ on $\partial\Omega$ is to seek a function in $\mathcal{S}$ that mimimizes $J(u)$. The functional J is said to be a *variational problem* for the differential equation $\Delta u = F$.]

Remark on notation for integrals. When integrating functions in dimensions between 1 and 3, people often use notations such as $\int f(x)dx$, $\iint f(x,y)dx\,dy$ and $\iiint f(x,y,z)dx\,dy\,dz$. However this notation scheme becomes unsustainable in higher dimensional situation. Instead one uses only one integral sign, and the integrals are written in the form $\int_D \alpha$, where D is the geometric body over which the integrand α is integrated. The integrand α usually are one of two forms: (a) $\alpha = f(x)\mu$, where μ is a "measure" (also called a "volume element" such as $d^n x$ when D is a bounded domain in $\mathbb{R}^n$, or (b) α is a "differential form", such as $f_1\,dx_2 \wedge dx_3 + f_2\,dx_3 \wedge dx_1 + f_3\,dx_1 \wedge dx_2$. The integrals of type (a) are *unsigned*, while those of type (b) *have signs*. The integrals with signed allows cancellation to happen, and the fundamental theorem of calculus, in the form $\int_{\partial D} \alpha = \int_D d\alpha$, holds.

3. Define explicitly integrals over spheres in $\mathbb{R}^3$, especially the unit sphere S^3 in $\mathbb{R}^4$, and compute the integral over S^3 of the inner product of a smooth vector field $\vec{F}$ on $\mathbb{R}^4$ with components (x,y,z,w) with the outer normal of S^3. Then verify the divergence theorem by computing the divergence of $\vec{F}$ Here x,y,z,w are the coordinates of $\mathbb{R}^4$.

(Motivation: The divergence theorem in dimension 4 is used on page 231 of Strauss. Strauss refers to A.1, but A.1 does not really discuss divergence theorem in dimension 4 or higher. A good book in this aspect is the thin book "Calculus on Manifods" by Spivak.)