Math 4250 Problem Set 7, Spring 2024

Part 1. From Strauss, *Partial Differential Equations.*

- Exercise 5.5, #2, page 145
- Exercise 6.1, #6, #7, #10, pages 160–161
- Exercise 6.1 (extra credit), #11, page 161
- Exercise 6.2, #3, page 165
- Exercise 6.3, #3, page 172

Part 2.

1. (extra credit) Let $f(x)$ be a continuous $\mathbb{Z}$-periodic $\mathbb{C}$-valued function on $\mathbb{R}$. Let

$$c_n = c_n(f) = \int_0^1 f(x) e^{-2\pi\sqrt{-1}nx} dx$$

be the Fourier coefficients of f. Assume that $\sum_{n\in\mathbb{Z}} |c_n|$ converges.

(a) Show that the infinite series $F(x) := \sum_{n\in\mathbb{Z}} c_n e^{2\pi\sqrt{-1}nx}$ converges for every x.

(b) Show that the function $F(x)$ defined by the convergent infinite series in (a) is continuous and $\mathbb{Z}$-periodic.

(c) Show that $F(x)$ has the same Fourier coefficients as $f(x)$.

(d) Conclude from the completeness of the family $\left(e^{2\pi\sqrt{-1}nx}\right)_{n\in\mathbb{Z}}$ in $C([0,1])$ proved in class that $f(x) = F(x)$ for all $x \in \mathbb{R}$.

2. Suppose that $u(x)$ is a twice differentiable function on $\mathbb{R}$ which satisfies the ordinary differential equation

$$u'' + b(x)u' - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0,1)$.

(a) Show that u cannot have a positive local maximum in the open interval $(0,1)$, that is, have a local maximum at a point p where $u(p) > 0$. Also show that u cannot have a negative local minimum in $(0,1)$.

(b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0,1]$.

(c) Give an explicit example of a function $u_1(x)$ which satisfies an ODE

$$u'' + b_1(x)u' - c_1(x)u = 0,$$

where b_1, c_1 are continuous functions on $\mathbb{R}$, and $u_1(x)$ does have a local maximum at a point $q \in (0,1)$ with $u_1(q) > 0$.

3. Let $u(x, y)$ be a twice continuously differentiable function on an open domain $\mathcal{D}$ in $\mathbb{R}^2$. Suppose that u satisfies the differential equation

$$4u_{xx} + 3u_{yy} - 5u = 0$$

in $\mathcal{D}$. Show that u cannot have a local positive maximum at a point of $\mathcal{D}$, i.e. a local maximum $p \in \mathcal{D}$ with $u(p) > 0$. Also show that u cannot have a local negative minimum at a point of $\mathcal{D}$.

4. Let $u(x, y)$ be a twice continuously differentiable function on a bounded open domain $\mathcal{D}$ of $\mathbb{R}^2$. Suppose that u satisfies the differential equation

$$4u_{xx} - 2u_{xy} + 3u_{yy} + 7u_x + u_y - 5u = 0.$$

in $\mathcal{D}$.

(a) (extra credit) Show that u cannot have a local positive maximum at a point of $\mathcal{D}$, i.e. a local maximum $p \in \mathcal{D}$ with $u(p) > 0$. Also show that u cannot have a local negative minimum at a point of $\mathcal{D}$.

[Hint: If $A = (a_{ij})$ and $B = (b_{ij})$ are positive semi-definite symmetric $n \times n$ matrices, then $\sum_{i,j=1}^{n} a_{ij} b_{ij} \geq 0$.]

(b) Suppose that $u(x, y)$ extends to a continuous function on the closure of $\mathcal{D}$ and that u vanishes on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$ of $\mathcal{D}$. Show that u is identically 0 on $\mathcal{D}$.