Math 4250 Problem Set 5, Spring 2024

Part 0. Read pages 92–98 of Strauss on Robin's boundary condition.

Part 1. From Strauss, *Partial Differential Equations*.

- Exercise 3.2., #10 page 10. Solve this problem in two different ways.
  - (a) Using the method in Strauss, section 3.2.
  - (b) Use the method of separation of variable and Fourier series expansion.
- Exercise 3.3, #2, page 71
- Exercise 3.4, #2, #4, #11, pp. 79–80
- Exercise 4.1, #1, #4 , page 89
- Exercise 4.2, #2, page 92
- (extra credit) Exercise 4.3, #8, page 101

Part 2.

1. Use Duhamel's principle (page 78 of Strauss) to find a formula for a particular solution of the ODE $u''(x) + a^2 u(x) = f(x)$, where $a > 0$ is a positive real number.

2. Solve the following variant of the heat equation

$$\Big(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} + bt^2\Big)u(x,t) = 0,$$

for a function $u(x,t)$ on the domain $\{(x,t) \in \mathbb{R}^2 : x \in \mathbb{R},\ t > 0\}$, which satisfies the initial condition $\lim_{t\to 0^+} u(x,t) = f(x)$ for a given function $f(x)$ on the real line, where $k, b > 0$ are positive constants.
[Hint: The solutions of the related ODE $w'(t) + bt^2 w(t) = 0$ are $c\,e^{-bt^3/3}$. So try making the change of variables $u(x,t) = e^{-bt^3/3}v(x,t)$, derive a equation for $v(x,t)$ and solve that equation for $v(x,t)$.]

3. (extra credit) Let $u_1(x), u_2(x)$ be smooth function on a finite interval $[a,b]$ which satisfies two second order linear ODE's

$$(a(x)u_i'(x))' + b(x)u_i(x) = \lambda_i u_i(x), \quad i = 1, 2,$$

where $a(x), b(x)$ are smooth functions on $[a,b]$, and λ_1, λ_2 are real constants.

(a) Suppose that $u_1(x), u_2(x)$ both satisfy the homogeneous Dirichlet boundary conditions $u_i(a) = 0 = u_i(b)$ for $i = 1, 2$. Show that

$$\int_a^b u_1(x)u_2(x)\,dx = 0.$$

(b) Find other boundary conditions so that the statement in (a) hold for any two solutions satisfying the boundary conditions you specify.