Math 4250 Problem Set 4, Spring 2024

Part 1. From Strauss, *Partial Differential Equations*, Chapter 1.

- Exercise 2.5, #4, page 56
- Exercise 3.1, #1, #4, page 60
- Exercise 3.2, #3, #10 page 66

Part 2.

1. Let C be the circle of radius 1 and let $u(\theta, t)$ be the temperature at a point $e^{\sqrt{-1}\theta}$ at time t. Thus we need that $u(\theta + 2\pi) = u(\theta)$. Suppose that $u(\theta, t)$ satisfies the heat equation $u_t = u_{\theta\theta}$. Let

$$E(t) = \frac{1}{2} \int_{-\pi}^{\pi} u^2(\theta, t)\, d\theta.$$

(a) Show that $E'(t) \leq 0$.

(b) Suppose that the initial temperature $u(\theta, 0) = 0$ for all θ. Show that $u(\theta, t) = 0$ for all $t \geq 0$ and all θ.

2. Suppose that a function $u(t)$ satisfies the differential equation

$$u'' + b(t)u' + c(t)u = 0 \tag{1}$$

on the interval $[0,\ A]$ and that the coefficients $b(t)$ and $c(t)$ are both continuous on $[0, A]$, so that $b(t)$ and $c(t)$ are bounded on $[0, A]$. Say $|b(t)| \leq M$ and $|c(t)| \leq M$. Define a function $E(t)$ on $[0, A]$ by

$$E(t) := \frac{1}{2}(u'^2 + u^2).$$

(a) Show that for there exists a positive constant γ (depending on M) such that $E'(t) \leq \gamma E(t)$ for all $t \in [0, A]$.

[Suggestion: use the simple inequality $2xy \leq x^2 + y^2$.]

(b) Show that $E(t) \leq e^{\gamma t} E(0)$ for all $t \in [0,\ A]$. [Hint: First use the previous part to show that $(e^{-\gamma t} E(t))' \leq 0$].

(c) Show that if $u(0) = 0$ and $u'(0) = 0$, then $E(t) = 0$ and hence $u(t) = 0$ for all $t \in [0,\ A]$. In other words, if $u'' + b(t)u' + c(t)u = 0$ on the interval $[0,\ A]$ and that the functions $b(t)$ and $c(t)$ are continuous, and if $u(0) = 0 = u'(0)$, then $u(t) = 0$ for all $t \in [0, A]$.

(d) Use (c) to prove the *uniqueness theorem*: if $v(t)$ and $w(t)$ both satisfy equation

$$u'' + b(t)u' + c(t)u = f(t) \tag{2}$$

and have the same initial conditions, $v(0) = w(0)$ and $v'(0) = w'(0)$, then $v(t) = w(t)$ for all $t \in [0,\ A]$.

(e) Assume the coefficients $b(t)$, $c(t)$, and $f(t)$ in equation (2) are periodic with period P, that is, $b(t+P) = b(t)$ etc. for all real t. If $\phi(t)$ is a solution of equation (2) that satisfies the *periodic boundary conditions*

$$\phi(P) = \phi(0) \quad \text{and} \quad \phi'(P) = \phi'(0), \tag{3}$$

show that $\phi(t)$ is periodic with period P: $\phi(t+P) = \phi(t)$ for all $t \geq 0$. Thus, the periodic boundary conditions (3) do imply the desired periodicity of the solution

(f) (extra credit) If we assume, instead of the continuity of $b(t)$ and $c(t)$, only that both $b(t)$ and $c(t)$ are bounded on $[0, A]$. Do the statements (a)-(e) above still hold? (Either give a proof, or a counter-example.)

3. Let $u(x,t)$ be the temperature at time t at the point x, $-L \leq x \leq L$, where L is a positive real number. Assume $u(x,t)$ is twice differentiable and satisfies the heat equation $u_t = u_{xx}$ for $0 < t < \infty$ with the boundary condition $u(-L,t) = u(L,t) = 0$ and initial condition $u(x,0) = f(x)$ for a function $f(x)$ on $[-L, L]$.

(a) Show that $E(t) := \dfrac{1}{2}\displaystyle\int_{-L}^{L} u^2(x,t)\,dx$ is a decreasing function of t.

(b) Use this to prove uniqueness for the heat equation with these specified initial and boundary conditions $u(-L,t) = f(t)$, $u(L,t) = g(t)$.

(c) Suppose that $u(x,0) = \varphi(x)$ is an even function of x, and $u(-L,t) = u(L,t)$ for all $t \geq 0$. Show that the temperature $u(x,t)$ at later times is also an even function of x.