Math 4250 Problem Set 3, Spring 2024

Part 1. From Strauss, *Partial Differential Equations*, Chapter 1.

- Exercise 2.2, #4, page 41
- Exercise 2.3, #1, #2, page 45
- Exercise 2.3, #6, page 46
- Exercise 2.4, #4, page 52
- Exercise 2.4, #11, page 53

Part 2.

1. (extra credit) Exercise 2.4, #16, page 54

2. (extra credit) This problem is about the asymptotic expansion of the complementary error function $\mathrm{Erfc}(x) := 1 - \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ as $x \to \infty$.

(a) Show that $\int_x^\infty e^{-t^2} dt = \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2} dt$.
(Hint: integrate by parts)

(b) Show that $\int_x^\infty e^{-t^2} dt \leq \frac{e^{-x^2}}{2x}$ for all $x > 0$.
(Hint: $-t^2 \leq -x^2 - 2x(t-x)$ for all $t \geq x$.)

(c) Show that for every integer $n \geq 1$, we have

$$\int_x^\infty e^{-t^2} dt = e^{-x^2} \cdot \left[\tfrac{1}{2x} - \tfrac{1}{2^2 x^3} + \tfrac{1 \cdot 3}{2^3 x^5} - \tfrac{1 \cdot 3 \cdot 5}{2^4 x^7} + \cdots + (-1)^{n-1} \tfrac{1 \cdot 3 \cdots (2n-3)}{2^n x^{2n-1}}\right] + R_n(x),$$

where

$$R_n(x) = (-1)^n \tfrac{1 \cdot 3 \cdots (2n-1)}{2^n} \int_x^\infty t^{-2n} e^{-t^2} dt.$$

Note that when $n = 1$, the product $\prod_{i=1}^{n-1} (2i-1)$ above is understood to be 1.

(d) Show that $\int_x^\infty t^{-2n} e^{-t^2} dt \leq \frac{e^{-x^2}}{2x^{2n+1}}$, for all integer $n \geq 0$ and all $x > 0$. In particular $|R_n(x)| \leq \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1} x^{2n+1}} e^{-x^2}$, for all $x > 0$.

(e) Show that the infinite series

$$\sum_{m \geq 1} (-1)^{m-1} e^{-x^2} \frac{\prod_{i=1}^{m-1} (2i-1)}{2^m x^{2m-1}}$$

diverges for every $x \neq 0$.

(f) The statements (c) and (d) says that for every positive integer n, the finite series $S_n(x) := \sum_{m=1}^{n} e^{-x^2} \frac{\prod_{i=1}^{m-1}(2i-1)}{2^n x^{n-1}}$ is a good approximation of $\int_x^\infty e^{-t^2} dt$ up to an error $R_n(x) = O(x^{-2n-1})$ for $x \gg 0$. On the other hand, the statement (e) says that the $\lim_{n\to\infty} S_n(x)$ does not exist, for every $x \neq 0$. Explain why these two statements do not contradict each other.