Math 4250 Problem Set 10, Spring 2024

Part 1. From Strauss, *Partial Differential Equations*.

- Exercise 10.2, #1, page 270. (Note on typo: should be "initial conditions (25)")
- Exercise 10.2, #2, #5, page 270

Part 2.

1. Show that

$$\int x^2 J_0(x)dx = x^2 J_1(x) + xJ_0(x) - \int J_0(x)dx$$

and

$$\int_0^1 xJ_0(\alpha x)dx = \frac{1}{\alpha} J_1(\alpha).$$

2. Suppose that α is a root of the equation $J_0(x) = 0$. Show that

$$\int_0^\alpha J_1(x)dx = 1.$$

3. Let $I_0(x)$ be the function $I_0(x) := J_0(\sqrt{-1}x)$.

(i) Show that $I_0(x)$ is a solution of the differential equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - xy = 0.$$

(ii) Consider the function $u(x) = \sqrt{x}\, I_0(x)$ on $(0, \infty)$. Show that $u(x)$ satisfies the differential equation

$$\frac{d^2u}{dx^2} = \left(1 - \tfrac{1}{4x^2}\right)u.$$

(iii) (extra credit) Show that

$$\lim_{x\to\infty} I_0(x) = \infty.$$

Note that (ii) and (iii) suggest that $u(x) \sim A\, e^x$ for some constant $A > 0$ as $x \to \infty$.

(iv) (extra credit) Let $v(x) := e^{-x}u(x)$. Show that $v(x)$ satisfies the differential equation

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + \frac{v}{4x^2} = 0,$$

and use this equation to obtain an asymptotic expansion of $v(x)$ in powers of $\frac{1}{x}$, assuming that $\lim_{x\to\infty} v(x) = A$ for some positive constant A.

6. (extra credit) Let $u(x)$ and $v(x)$ be solutions of ordinary differential equations

$$u'' + a(x)u = 0, \qquad v'' + b(x)v = 0,$$

on $(0, \infty)$, where $a(x)$ and $b(x)$ are continuous functions on $(0, \infty)$.

(a) Show that

$$\int_\alpha^\beta (b - a)uv\, dx = (vu' - uv')\Big|_\alpha^\beta$$

for any $0 < \alpha < \beta$.

(b) Suppose that α and β are consecutive zeroes of u and that $u(x) > 0$ in the interval $\alpha < x < \beta$. If $a(x) \leq b(x)$ in this interval, show that $v(x)$ must be zero somewhere in this interval by using that $u'(\alpha) > 0$ and $u'(\beta) < 0$ and there is a contradiction unless v is zero somewhere in this interval. This is the *Sturm oscillation theorem.*

(c) In the special case where $a(x) = b(x)$ and $u(x)$ and $v(x)$ are linearly independent solutions of the same equation, conclude that between any two zeroes of u there is a zero of v, and vice versa. Thus the zeroes of u and the zeros of v interlace. A special case of this is the interlacing of the zeroes of $\sin x$ and $\cos x$.

(d) Suppose that $b(x) \geq c^2 > 0$ for some constant c. Show that $v(x)$ must have infinitely many zeroes by comparing v with a collation of $u'' + c^2 u = 0$.

(e) Show that every solution of $v'' + (1 - \frac{1}{x^2} v = 0$ must have infinitely many zeroes .