Some old exam problems, April 15, 2023

1. Prove that for any complex number z not equal to a negative integer,

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

Pay particular attention to the constants involved.

2. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}.$$

3. Show that the function

$$f(z) := \sum_{n=1}^{\infty} e^{-n} \sin nz$$

is holomorphic in the strip $-1 < \mathrm{Im}(z) < 1$.

4. Show that $F(z) = \sin \pi z$ is the only entire function which satisfies the following properties:

   1. $F(z) = F(z+1)$ for all $z \in C$.
   2. $F(0) = 0$ and $F(\frac{1}{2}) = 1$.
   3. $|F(x+iy)| \le e^{\pi|y|}$ for all $x, y \in \mathbb{R}$.

5. Show that the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$ is **NOT** conformally equivalent to the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

6. For each of the following statements, mark whether they are true or false. No work needs to be shown for this problem. (5 problems, 4 points each)

   1. The winding number is always an integer.
   2. A non-zero holomorphic function $f(z)$ can have a distinct sequence $z_n \to z_0$ and $f(z_n) = 0$.
   3. A non-zero holomorphic function $f(z)$ can have a distinct sequence $z_n \to z_0$ and $f(z_n) = 0$.
   4. A bounded entire function must be constant.

5. Nearby an essential singularity z_0, a holomorphic function (except at z_0) takes on every value in the complex plane.

6. A meromorphic function can have only isolated zeroes and poles.

7. Answer the following questions.

   1. What is a Laurent series expansion? State it carefully.

   2. Find the Laurent series expansion in powers of z around $z = 0$ for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

   and say where the series converges.

   3. Find the Laurent series expansion in powers of z around $z = 0$ for the function

$$g(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

   and where does the Laurent expansion for $g(z)$ converge?

8.

   1. Give the full general definition of a residue (including how to calculate it).

   2. State completely and precisely the Cauchy Residue Theorem.

   3. Calculate all the residues of

$$f(z) = \frac{z+1}{z^2 - 2z}$$

   4. Calculate the following integral

$$\int_{|z|=\rho} \frac{z+1}{z^2-2z}\, dz,$$

   both when $0 < \rho < 1$ and when $1 < \rho < \infty$.

9.

   (a) (5 points) State Rouché's theorem.

   (b) (10 points) How many zeros (counting multiplicities) does the function $f(z) = e^z - 3z^2$ have inside the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$? Justify your answer.

   (c) (5 points) How many **distinct** zeros (i.e. zeroes of multiplicity 1) does $f(z) = e^z - 3z^2$ have inside the unit circle? Justify your answer.

10.

(a) (5 points) State Liouville's Theorem.

(b) (10 points) Prove or disprove: The image of $\mathbb{C}$ under a nonconstant entire mapping is dense in $\mathbb{C}$. (Hint: The mapping having an image that is dense in $\mathbb{C}$ means that the values of the mapping come arbitrarily close to every complex number. Recall how the Casorati-Weierstrass Theorem was proven.)

11.

(a) (5 points) Give the definition of a residue.

(b) (5 points) Determine and classify all singularities of

$$f(z) = z^3 e^{\frac{1}{z}} + \frac{z^2}{(z-1)^3}.$$

(c) (10 points) Compute the residues of $f(z)$ at the singularities you found in part (b). (Hint: What is the expansion of $e^{\frac{1}{z}}$ formally? It might be easier computing these **DIRECTLY** from the definition of a residue.)