# Some old exam problems, March 15, 2023

1.(a) Find all complex zeros of $\cos z - \sin z$.

(b) What is the radius of convergence of the power series of

$$f(z) = \frac{1}{\cos z - \sin z}$$

around $z = 0$? Justify your answer. (Hint: **DO NOT** expand this into a power series. You should be able to see the radius without any computation.)

2. Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

3. Find all zeros of the rational function

$$\frac{z^3 - 7z^2 + 15z - 9}{z^2 + 3z - 4}$$

and determine their orders.

4. Show that

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx = \frac{\pi \log a}{2a} \text{ when } a > 0.$$

5. Identify all zeros, poles, removable singularities, and/or essential singularities. For zeros and poles, give the orders. Be sure to consider the point at $\infty$. You do not need to compute residues.

$$f(z) := \frac{1}{z} \sin\left(\frac{\pi z}{1 - z}\right)$$

6. Determine all possible values of

(a) $\log(i)$

(b) $\sqrt{i}$

(c) $(3i)^i$

(d) $5^i$

6. Let $p_n$ be the degree $n$ Taylor polynomial of $e^z$, i.e., $p_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!}$.

(a) For any $n \geq 1$ and any $k$ between 1 and $n$, show that

$$\frac{|z|^{n-k}}{(n-k)!} \leq \frac{|z|^n}{n!}\left(\frac{n}{|z|}\right)^k.$$

Conclude that $p_n(z)$ has $n$ zeros on the disk $|z| \leq 2n$ for each positive integer $n$.

(b) Show that for any $n \geq 1$

$$\frac{|z|^n}{n!} < \delta^n e^{\delta^{-1}|z|} \text{ for any } \delta > 0.$$

Prove that there is a $\delta_0 > 0$ such that, for all $n \geq 1$, none of the zeros of $p_n$ lie in the disk $|z| \leq \delta_0 n$.

Hint: Don't worry about the exact value of $\delta_0$—just prove the statement for $\delta_0$ sufficiently small.

7. Suppose the Taylor series for $\tan z$ is given by $\tan z = \sum_{n=0}^{\infty} c_n z^n$ for some constants $c_n$.

(a) Prove that

$$\frac{1}{z-\frac{\pi}{2}} + \frac{1}{z+\frac{\pi}{2}} + \tan z$$

is holomorphic on the disk $|z| < \frac{3\pi}{2}$ (in particular, show that the apparent singularities are removable).

(b) Conclude that for any $\delta > \frac{2}{3\pi}$, there is a finite constant $C_\delta$ such that

$$\left| c_n - \frac{2^{n+2}}{\pi^{n+1}} \right| \leq C_\delta \delta^n$$

whenever $n$ is odd. This means that the odd coefficients in the Taylor series for $\tan z$ are equal to $2^{n+2}\pi^{-n-1}$ up to an error which is proportionally much smaller than this as $n \to \infty$. Hint: Use the Cauchy inequalities.

8. For each of the following statements, mark whether they are true or false.

(a) The winding number is always an integer.

(b) A non-zero holomorphic function $f(z)$ can have a distinct sequence $z_n \to z_0$ and $f(z_n) = 0$.

(c) A bounded entire function must be constant.

(d) Nearby an essential singularity $z_0$, a holomorphic function (except at $z_0$) takes on every value in the complex plane.

(e) A meromorphic function can have only isolated zeroes and poles.