Math 4100 Homework 9, Spring 2023

Part 1. From Ash–Novinger, *Complex Variables*.

- Ch. 4, pp. 24–25, #2, #6
- Ch. 4, pp. 30–31, #1
- Ch. 6, p. 5, #5

Part 2.

(1) Suppose that $f : \mathbb{C} \to \mathbb{C}$ a conformal automorphism of $\mathbb{C}$, i.e. f is holomorphic, one-to-one and onto.

  (a) The great Picard's theorem asserts that for every entire function on $\mathbb{C}$ with essential singularity at ∞, there exists a point $z_0 \in \mathbb{C}$ such that for every $z \in \mathbb{C} \smallsetminus \{z_0\}$, the inverse image $h^{-1}(z)$ of z is in infinite set. Use this fact to show that f is meromorphic at ∞.

  (b) Prove that f is induced by a unique linear fractional transformation.

(2) (extra credit) Let $a \neq b$ be two complex numbers. Consider linear fractional transformations of the form

$$w = T_k(z) = k \cdot \frac{z-a}{z-b}, \quad k \neq 0, \ k \in \mathbb{C}.$$

  (i) Show that circles on the z-plane passing through a and b are inverse images under T_k of straight lines through the origin of the w-plane.

  (ii) Show that for each positive real number $\alpha > 0$, the set

$$\{z \in \mathbb{C} : |z-a| = \alpha \cdot |z-b|\}$$

  is a circle on the z-plane if $\alpha \neq 1$ and is the line orthogonal to the line segment $\overline{a\,b}$ passing through the midpoint of $\overline{a\,b}$. Moreover it is the inverse image under T_k of the circle

$$|w| = \alpha \cdot |k|$$

  on the w-plane.

  (iii) The two families of circles, $\{C_1\}$ in (i) $\{C_2\}$ and (ii) above are called the *Steiner circles* determined by a and b. Show the following.

  * Every C_1 meets every C_2 in right angles.
  * There is exactly on C_1 and one C_2 passing through any given point on the z-plane.

* Each reflection about a C_1 transforms every C_2 to itself and preserves the family $\{C_1\}$. Each reflection about a C_2 transforms every C_1 to itself and preserves the family $\{C_2\}$.

(3) (extra credit) Let T be the linear fractional transformation attached to an invertible 2×2 matrix A.

(i) Show that T has two distinct fixed points $a \neq b \in \hat{\mathbb{C}}$, i.e. $T(a) = a$ and $T(b) = b$ if and only if A is diagonalizable over $\mathbb{C}$. If so, show that there exists a non-zero complex number $0 \neq k \in \mathbb{C}$ such that $w = T(z)$ satisfies

$$\frac{w-a}{w-b} = k \cdot \frac{z-a}{z-b}.$$

(Such a linear fractional transformation is said to be *hyperbolic* if $k \in \mathbb{R}$, *elliptic* if $|k| = 1$.) Note that T can be both hyperbolic and elliptic, in which case T^2 is the identity map on $\hat{C}$.)

(ii) Show that T has exactly one fixed point if and only if the linear fractional transformation A is not diagonalizable over $\mathbb{C}$.

(Such a linear fractional transformation is said to be *parabolic*. A linear fractional transformation which is neither hyperbolic, elliptic, nor parabolic is said to be *loxodromic*.)