Math 4100 Homework 8, Spring 2023

Part 1. From Ash–Novinger, *Complex Variables.*

- Ch. 4, p. 17, #1, #3
- Ch. 4, pp. 20–21, #2, #4

Part 2.

(1) Let $\tau : \mathbb{C} \to \mathbb{C}$ be the complex conjugation $z \mapsto \bar{z}$.

(a) Let $\gamma_1 : [0,1] \to \mathbb{C}$ and $\gamma_2 : [0,1] \to \mathbb{C}$ be two smooth curves on $\mathbb{C}$ such that $\gamma_1(0) = \gamma_2(0)$. Show that the angle at $\gamma_1(0)$ between the two curves γ_1 and γ_2 is equal to the angle at $\overline{\gamma_1(0)}$ between $\tau \circ \gamma_1$ and $\tau \circ \gamma_2$.

(b) Prove that τ is not equal to a linear fractional transformation. (Note: you need to show that for every invertible 2×2 matrix A with complex coefficients, τ is not equal to the linear fractional transformation attached to A.)

(2) (extra credit) For every positive integer n, denote by C_n the square whose vertices are $(n + \frac{1}{2})(\pm 1 \pm \sqrt{-1}$. Prove that there exists a constant $M > 0$ such that

$$|\cot(\pi z)| \leq M \quad \text{and} \quad |\csc(\pi z)| \leq M$$

for all positive integer n.

(3) (extra credit) We keep the notation in problem 2 above, and consider the integral

$$\oint_{C_n} dz,$$

where $a \in \mathbb{R} \setminus \mathbb{Z}$ is a real number and is not an integer.

(a) Find all singularities of the meromorphic function $\frac{\cot(\pi z)}{(a+z)^2}$ on $\mathbb{C}$ and compute the residues at the singularities.

(b) Calculate the sum of the infinite series

$$\sum_{-\infty}^{\infty} \frac{1}{(a+n)^2}\,.$$