Math 4100 Homework 4, Spring 2023

Part 1. From Ash–Novinger, *Complex Variables.*

- Ch. 2, pp. 26–28, #7, #11, #19, #24

Part 2.

(1) Find the Taylor series expansion of the rational function

$$f(z) = \frac{1+2z}{1-z^3}$$

explicitly.

(2) Suppose that $f(z)$ is a complex analytic function on the whole complex plane, and there is a sequence of *non-zero* complex numbers $a_0, a_1, a_2, \ldots$ such that

$$f(z + a_i) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

Is $f(z)$ a constant function? Either prove this statement, or give a counter-example.

(3) (extra credit) For what values of z is the series

$$\sum_{n=0}^{\infty} \Big(\frac{z}{1-z}\Big)^n$$

convergent?

Some problems from past exams, *not* part of the homework assignment. (These are among the harder one.)

1. (a) Write down a fractional linear transformation $S(z)$ that maps $-i$, 0, and i into 0, i, and $2i$ respectively. (If you wish, you may leave the answer as a composition of functions – no need to simplify.)

(b) Is the transformation from part (a) the unique one with such properties? Justify your answer.

2. (a) Write down a fractional linear transformation $S(z)$ that maps $-i$, 0, and i into 0, i, and $2i$ respectively. (If you wish, you may leave the answer as a composition of functions – no need to simplify.)

Is the transformation from part (a) the unique one with such properties? Justify your answer.

3. Let Ω be an open disk, and suppose that f and g are holomorphic functions on Ω which satisfy the equality

$$\Re(f(z)) = \Re(g(z)) \text{ for all } z \in \Omega.$$

Prove that there is a *real* constant c such that

$$f(z) = g(z) + ic \text{ for all } z \in \Omega.$$

4. (a) Find the complex zeros of $\cos z - \sin z$.

(b) What is the radius of convergence of the power series of

$$f(z) = \frac{1}{\cos z - \sin z}$$

around $z = 0$? Justify your answer. (Hint: **DO NOT** expand this into a power series. You should be able to see the radius without any computation.)

5. Let $P(z) = a_0 + a_1 z + \cdots + a_k z^k$ be a complex polynomial. Suppose $|P(z)| \leq 1$ whenever $|z| \leq 1$. Show that $|a_n| \leq 1$ for all $n = 0, 1, \ldots, k$. (Hint: Apply Cauchy's Representation Formula for derivatives.)