Math 4100 Homework 11, Spring 2023

Part 1. From Ash–Novinger, *Complex Variables.*

- Ch. 4, p. 45, #1. You are asked to show directly from basic properties of holomorphic functions that there can be no open subset U of $\mathbb{C}$ which contains the open unit circle $D(0,1)$ and a point z with $|z|=1$ and a holomorphic function g on U which coincides with f on $D(0,1)$.

Part 2.

(1) (a) Show that the infinite product

$$\prod_{n=1} (1-\frac{z}{n})e^{z/n}$$

converges uniformly on compact subsets of $\mathbb{C}$ and defines an entire function $f(z)$.

(b) (extra credit) Determine whether $|f(z)|$ is bounded on vertical strips of the form $\{z \in \mathbb{C} \mid a \le \mathrm{Re}(z) \le b\}$, where a, b are real numbers with $a < b$.

(2) Show that

$$\lim_{n\to\infty} \int_0^n u^{z-1}(1-\tfrac{u}{n})^{n+w-1} du = \int_0^\infty u^{z-1}e^{-u} du$$

for all z, w with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$. (This is the last step in the proof of $B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ which we discussed in class but did not carry out the details. Note that the limit at the left hand side involve n at both the upper limit of integration and also the integrand. For a fixed z, the integrand goes to $u^{z-1}e^{-u}$ as n goes to ∞.)

(3) (This problem gives an alternative proof of the formula $B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$.)

Show that for $z, w \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$,

$$\int_0^\infty e^{-x}x^{z-1}dx \cdot \int_0^\infty e^{-y}y^{w-1}dy = \int_0^\infty e^{-\xi}\xi^{z+w-1}d\xi \cdot \int_0^1 \eta^{z-1}(1-\eta)^{w-1}d\eta\,.$$

(Hint: Use the change of variables $x + y = \xi$, $y = \xi\eta$. Please be careful about convergence of integrals involved.)

(4) Prove that the infinite series

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

converges uniformly on the set

$$D_M := \{z \in \mathbb{C} \mid -1 \le \mathrm{Re}(z) \le 1,\ |\mathrm{Im}(z)|_M$$

for every $M > 0$, and

$$\lim_{M\to\infty} \sup_{z\in D_M} \left| \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \right| = 0.$$

(Note: This is a crucial part in the proof of $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ discussed in class.)

(5) (extra credit) Prove that

$$\cos(\pi z) = \prod_{n=0}^{\infty} \left[1 - \left(\frac{2z}{2n+1} \right)^2 \right].$$