Math 4100 Homework 10, Spring 2023

Part 1. From Ash–Novinger, *Complex Variables.*

- Ch. 6, p. 5, #6
- Ch. 6, p. 9, #1(c), #2

Part 2.

1. What are the residues of $\Gamma(z)$ at its poles?
2. (a) Prove that $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ for all $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

   (b) Prove that

$$|\Gamma(\sqrt{-1}y)|^2 = \frac{\pi}{y\,\sinh(\pi\,y)}, \quad |\Gamma(\tfrac{1}{2} + \sqrt{-1}y)|^2 = \frac{\pi}{\cosh(\pi\,y)}$$

   for all $y \in \mathbb{R}$.

3. (extra credit) Show that the improper integrals

$$\int_0^\infty \sin(x^2)\,dx, \quad \int_0^\infty \cos(x^2)\,dx$$

   converge, and evaluate them.

4. (extra credit) This problem leads to another integral representation of $\Gamma(z)$. Fix a positive number δ, and let $C_{\rho,\epsilon}$ be a contour which starts from a point $\rho > 0$ on the real axis, goes up to $\rho + \sqrt{-1}\epsilon$ for a small positive number ϵ, then goes to the left along a horizontal line parallel to the x-axis, circle the origin counter-clockwise, then go to the right along a horizontal line parallel to the x axis to reach $\rho - \sqrt{-1}\epsilon$, and finally back to ρ. Let C be a "limit contour" of $C_{\rho,\epsilon}$ as $\rho \to \infty$. Show that

$$\Gamma(w) = -\frac{1}{2\sqrt{-1}\sin(\pi w)} \int_C (-z)^{w-1} e^{-z}\,dz,$$

   where $(-z)^{w-1} = e^{(w-1)\log(-z)}$, with $-\pi < \arg(-z) \leq \pi$ and $\log(-z)$ takes real values for z on the negative real axis.

   (Hint: Compare the values of $(-z)^{w-1}$ on "upper horizontal part" and the "lower horizontal part" of $C_{\rho,\epsilon}$, estimate the integral over the small circle of radius ϵ, and use the integral representation of $\Gamma(w)$ when $\text{Re}(w) > 0$.)