Examples of Group Actions

Notes for Math 370
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In each of the following examples we will give a group G operating on a set S. We will describe the orbit space $G\backslash S$ in each example, as well as some stabilizer subgroups $\mathrm{Stab}_G(x)$ for elements $x \in S$. Often we can find a subset $F \subseteq G$ of G such that the composition $F \to S \to G\backslash S$ of the inclusion of $F \hookrightarrow S$ and the canonical projection $S \to G\backslash S$ is a bijection; we say that F is a *fundamental domain* if this is so. Also, recall that for each $x \in S$, there is a natural bijection from the coset $G/\mathrm{Stab}_G(x)$ to the orbit $G \cdot x$.

1. Let $G = \mu_2 = \{1, -1\}$, $S = \mathbb{R}$, and the action is induced by multiplication of real numbers. The for each $0 \neq x \in \mathbb{R}$, we have $\mathrm{Stab}_G(x) = \{1\}$, and the G-orbit of x is $\{x, -x\}$. On the other hand, $\mathrm{Stab}_G(0) = G$, and the G-orbit of $0 \in S$ is $\{0\}$. The subset of all non-negative real numbers form a fundamental domain F.

2. Let n be a positive integer, let $G = \mathrm{GL}_n(\mathbb{C})$ and let $S = M_n(\mathbb{C})$. In this example we consider the left action of G on S given by matrix multiplication. That is, for $g \in G$ and $A \in S$ considered as two $n \times n$ matrices, the action is $(g, A) \mapsto g \cdot A$. Considering this action is essentially equivalent to the problem of row reduction for $n \times n$ matrices.

   There is one *big* orbit, consisting of all invertible $n \times n$ matrices; the stabilizer of each element in this orbit is trivial. There is exactly one fixed point of G, namely the zero matrix. A fundamental domain is given by the set of all $n \times n$ matrices in *row reduced echelon form*. Here a matrix is said to be in *row reduced echelon form* if for each row of this matrix, the first non-zero entry in this row is equal to 1, this "leading term" 1 is the only non-zero entry in the column it occupies, and the distance of the leading term to the left edge of the matrix increases as we go down the rows. It is allowed for some rows to be zero; such zero rows has to "pile up at the bottom". Here are some examples of such matrices in the case when $n = 3$:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&a\\0&1&b\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}1&a&0\\0&0&1\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&a&b\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&1&a\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

   In fact these gives all row reduced matrices in echelon form when $n = 3$.

3. Again let n be a positive integer, $G = \mathrm{GL}_n(\mathbb{C})$, $S = M_n(\mathbb{C})$, just as in example 2 above. However the action we consider now is given by conjugation: For $g \in G$ and $A \in S$ considered as two $n \times n$ matrices, the action is $(g, A) \mapsto g \cdot A \cdot g^{-1}$. The problem of

determining the orbit of this action is essentially this: Given a square matrix A, find a "simplest" representative in the conjugacy class of A. Of course this is strongly related to the theory of eigenvalues, eigenvectors and diagonalization. A complete answer is given by the theory of *Jordan canonical forms*, which will be discussed in Math 371 using the structure theory of modules over principal ideal domains. In the following we give a partial description of the set of all G-orbits.

A G-orbit here is nothing other than the $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of an $n \times n$ matrix A with coefficients in $\mathbb{C}$. All elements in this orbit have the same characteristic polynomial, given by $\mathrm{char}(A) := \det(X \cdot \mathrm{Id} - A)$. It is easy to see that every monic polynomial in $\mathbb{C}[X]$ of degree n is the characteristic polynomial of some element in $M_n(\mathbb{C})$. So we have a surjection $\mathrm{char_n}$ from $G\backslash S$ to the space $\mathrm{Monic}_n(\mathbb{C})$ of all monic polynomials of degree n with coefficients in $\mathbb{C}$, given by the characteristic polynomial. In general this is not a bijection, but finite-to-one: for any monic polynomial $f(X)$ of degree n, there are a finite number of $\mathrm{GL}_n(\mathbb{C})$-conjugacy classes with $f(X)$ as the characteristic polynomial. If $f(X)$ has n distinct roots, any $A \in M_n(\mathbb{C})$ with $f(X)$ as its characteristic polynomial is diagonlizable, hence $\mathrm{char}_n^{-1}(f(X))$ consists of just one element. On the other hand, if $f(X)$ has multiple roots, then $\mathrm{char}_n^{-1}(f(X))$ has more than one elements. For instance, if $n = 2$ and $f(X) = (X - \lambda)^2$ for some $\lambda \in \mathbb{C}$, then ${\mathrm{char}_2}^{-1}(f(X))$ has two elements, represented by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ respectively.

4. Let G be a group. The group law of G gives a left action of G on $S = G$. This action is usually referred to as the left translation. This action is transitive, i.e. there is only one orbit. The stabilizer subgroups are all trivial.

   Let H be a subgroup of G. The group law of G defines a left action of H on $S = G$; this is the restriction of the previous action to the subgroup H. An H-orbit is exactly a right H-coset, and the orbit space for this action is exactly the space $H\backslash G$ of all right H-cosets in G. (We designed the notations so that the two meanings of $H\backslash G$, one for the H-cosets and the other for the H-orbits, coincide.)

5. As in example 4 above, let G be a group and let $S = G$. Consider the conjugation action: $g \in G$ sends $x \in G$ to $g\,x\,g^{-1}$. The orbits of are simply the conjugacy classes in G. The stabilizer subgroup of $x \in G$ is just the centralizer subgroup $\mathrm{Z}_G(x)$ of x in G, consisting of all elements of G which commute with x; it is equal to the whole group G if and only if x is in the center of G. For a subgroup H of G, its stabilizer subgroup $\mathrm{Stab}_G(H)$ with respect to the conjugation action is called the normalizer subgroup of H in G, denoted by $\mathrm{N}_G(H)$. A subgroup H of G is normal if and only if $\mathrm{N}_G(H) = G$.

   The orbit space decomposition of S in this case is just the decomposition of G into its conjugacy classes. This is a very useful tool for studying finite groups. For instance when the cardinality of G is a power of p, one conclude from the decomposition of G into conjugacy classes that the center of G is not the trivial subgroup if G is not trivial.

6. Let $G = \mathbb{Z}^2$, $S = \mathbb{R}^2$, and the action of G on S is given by vector addition. In this case the stabilizer subgroup of every point of S is trivial. The subset $[0,1) \times [0,1)$ of $\mathbb{R}^2$ is a fundamental domain. The orbit space $\mathbb{Z}^2 \backslash \mathbb{R}^2$ can be visualized as follows. Start with the square $[0,1] \times [0,1]$. First, identify the top edge and the bottom edge; imagine that you glue these two edges together to form a cylinder. Then you identify the other two edges of the square; in the previous mental picture of the cylinder, you glue together the two rims of the cylinder. Thus we see that $\mathbb{Z}^2 \backslash \mathbb{R}^2$ represents a *torus*, shaped like a donut.

7. Let S be one of the Platonic solids and G be the group of all symmetries of S. One of the previous homework problems deals with the case when S is a regular n-gon and G is the dihedral group D_{2n}. This is the "easy" case. The more interesting cases are discussed in detail in Artin's book.