Notes on Bessel functions

The Bessel function $J_n(x)$, $n \in \mathbb{N}$, called the *Bessel function of the first kind of order n*, is defined by the absolutely convergent infinite series

$$J_n(x) = x^n \sum_{m\geq 0} \frac{(-1)^m\, x^{2m}}{2^{2m+n}\, m!\,(n+m)!} \qquad \text{for all } x \in \mathbb{R}\,. \tag{1}$$

It satisfies the Bessel differential equation

$$x^2\, J_n''(x) + x\, J_n'(x) + (x^2 - n^2) J_n(x) = 0\,. \tag{2}$$

The Bessel functions most relevant to this course are $J_0(x)$ and the closed related function $J_1(x)$. The function $J_0(x)$ is an even function, while $J_1(x)$ is odd; similarly for other $J_n(x)$'s, depending on the parity of n. We have

$$J_0'(x) = -J_1(x), \qquad J_1'(x) = J_0(x) - \frac{1}{x}\, J_1(x)\,. \tag{3}$$

Using the differential equations (3) and (2), it is not difficult to show that

$$\int x\, J_0^2(\alpha x)\, dx = \frac{x^2}{2}[J_0^2(\alpha x) + J_1^2(\alpha x)] + \text{Const.} \tag{4}$$

for all $\alpha \in \mathbb{R}$, and

$$(\beta^2 - \alpha^2) \int x\, J_0(\alpha x) J_0(\beta x)\, dx = x\, [\alpha\, J_0'(\alpha x) J_0(\beta x) - \beta\, J_0'(\beta x) J_0(\alpha x)] + \text{Const.} \tag{5}$$

for all $\alpha, \beta \in \mathbb{R}$. From (5) and (4) one deduces

$$\int_0^1 x\, J_0(\alpha x) J_0(\beta x)\, dx = 0 \tag{6}$$

if $J_0(\alpha) = J_0(\beta) = 0$, $\alpha, \beta > 0$, and $\alpha \neq \beta$. Moreover

$$\int_0^1 x\, J_0^2(\alpha x)\, dx = \frac{1}{2}[J_0^2(\alpha) + J_1^2(\alpha)] \tag{7}$$

if $J_0(\alpha) = 0$.

Exercise 1. Verify the equations (4), (5).

Exercise 2. Use the equation (2) to show that if α is a repeated root of $J_0(x)$ (i.e. $J_0(\alpha) = J_0'(\alpha) = 0$), then $J^{(n)}(\alpha) = 0$ for all $n \geq 0$. Conclude that $J_0(x)$ has no multiple root. (Hint: If α is a multiple root, the Bessel differential equation implies that the second derivative of $J_0(x)$ vanishes at α. Differentiate the Bessel differential equation, use it to conclude that the third derivative of the Bessel differential equation vanishes at α. Similarly for higher order derivatives.)

For large values of x, $J_n(x)$ behaves like a damped harmonic oscillator:

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{n\pi}{2} - \frac{\pi}{4})\,, \tag{8}$$

in the sense that

$$\lim_{x\to\infty} \frac{\sqrt{\frac{2}{\pi x}} \cos(x - \frac{n\pi}{2} - \frac{\pi}{4})}{J_n(x)} = 1\,.$$

Exercise 3. Use Maple to demonstrate the following statements.

(a) For large values of $x \in \mathbb{R}$, the *envelope* of $J_0(x)$ is

$$\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4}\right)$$

(Use several frames with different ranges of (large) values of x.)

(b) For large values of $x \in \mathbb{R}$, the difference of consecutive zeroes of $J_0(x)$ is close to π.

(c) The zeroes of $J_1(x)$ *interlace* with the zeroes of $J_0(x)$.