Practice problems before the final exam and some answers
Math 241: Calculus IV–001 fall 2000

Snap shot of the cover page of the final exam looks like

Final examination, Math 241: Calculus IV
December 15, 2000, 11:00AM–1:00PM

No books, papers, calculators or electronic device may be used, other than a hand-written note sheet at most $5'' \times 7''$ in size.

This examination consists of eleven multiple-choice questions and two long-answer questions. The multiple-choice questions are worth seven points each, with no partial credit. The correct and most appropriate answer to a multiple-choice question will be, in each case, just one of the seven choices (A), (B), (C), (D), (E), (F), or (G). Answer all multiple-choice questions on the answer sheet, which is page 14 of this exam. Only the answers on the answer sheet will be considered for grading.

The long-answer questions are worth twelve points each. You must **show all your work** and **box your answers**. Partial credits will be given only when a substantial part of a problem has been worked out. Merely displaying some formulas is not sufficient ground for receiving partial credit.

- Your name, printed:
- Your lecture section (circle one of the following):

Chai Shatz

1 to 11	12	13	Total (out of 101)

1. Let $f(x)$ be a continuous periodic function of period 1, which has a complex Fourier series expansion of the form

$$\sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i\, n\, x}\,.$$

Suppose that $f(x) + f(-x) = 2\cos(2\pi x)$ for all x. Which of the following statements is true?

A. $c_1 = 0$

B. $c_n = c_{-n}$ for all $n \in \mathbb{Z}$

C. $c_n + c_{-n} = 0$ for all $n \in \mathbb{Z}$

D. $\sum_{n\in\mathbb{Z}} (-1)^n\, c_n = 0$

E. $\sum_{n\in\mathbb{Z}} c_n = 0$

F. $c_0 = 0$

G. None of the above.

Ans. F. In fact $c_n + c_{-n} = 0$ if $n \neq \pm 1$, $c_1 + c_{-1} = 1$

2. Which of the following statements is true?

A. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{ire^{i\theta}} = 0$

B. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{-ire^{i\theta}} = 0$

C. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{re^{i\theta}} = 0$

D. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{-re^{i\theta}} = 0$

E. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{r^2e^{2i\theta}} = 0$

F. $\lim\limits_{\substack{r\to\infty \\ -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}}} e^{-r^2e^{2i\theta}} = 0$

G. None of the above.

Ans. D.

3. For every positive real number R, let

$$C_R = \{z \in \mathbb{C} : |z| = R,\ \operatorname{Im}(z) \leq 0\}$$

be the lower part of the circle $\{|z| = R\}$, from $-R$ to R, on the complex plane. Consider the limit

$$\lim_{R\to\infty} \oint_{C_R} \frac{e^{iz}}{z^2+1} dz\,.$$

Which of the following statements is true?

A. The limit is equal to 0.

B. The limit is equal to $\frac{e^{-1}}{2}$.

C. The limit is equal to 2π.

D. The limit is equal to $-e$.

D. The limit is equal to $e^{-1} - e$.

F. The limit is equal to e^{-1}.

G. None of the above.

Ans. D.

4. Let $f(x)$ be the function on $\mathbb{R}$ such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$, and

$$f(x) = e^x \quad \text{if } \pi \leq x \leq \pi\,.$$

Let $\sum_{n\in\mathbb{Z}} c_n\, e^{inx}$ be the complex Fourier series for $f(x)$. Which of the following statements is true?

A. Each c_n is a real number.

B. $c_0 = 0$

C. $\lim_{n\to\infty} c_n = 1$

D. $c_n \notin \mathbb{R}$ for each $n \in \mathbb{Z}$

E. $c_1 = \frac{1}{2\pi}$

F. $\sum_{n\in\mathbb{Z}} c_n = 1$

G. None of the above.

Ans. F

5. Suppose that $f(t)$ is a piecewise smooth function on $\mathbb{R}_{\geq 0}$ such that its Laplace transform $\mathcal{L}\{f(t)\}(s)$ is equal to $\dfrac{2\,e^{-3s}}{(s-2)^2}$. Which of the following statements is true?

A. $\lim_{t\to 1^+} f(t) = 0$

B. $\lim_{t\to 1^+} f(t) = e^3$

C. $\lim_{t\to 2^+} f(t) = e^{-2}$

D. $\lim_{t\to 2^+} f(t) = -e$

E. $\lim_{t\to 4^+} f(t) = 1$

F. No such function $f(t)$ exists.

G. None of the above.

Ans. A.

6. Suppose that $f(t)$ is a piecewise smooth function on $\mathbb{R}_{\geq 0}$ such that its Laplace transform $\mathcal{L}\{f(t)\}(s)$ is equal to $\dfrac{e^s}{s-1}$. Which of the following statements is true?

A. $\lim_{t\to 1^+} f(t) = 0$

B. $\lim_{t\to 1^+} f(t) = e$

C. $\lim_{t\to 2^+} f(t) = 2$

D. $\lim_{t\to 2^+} f(t) = e$

E. $\lim_{t\to 4^+} f(t) = e$

F. No such function $f(t)$ exists.

G. None of the above.

Ans. E

7. Which of the following is *not* the Laplace transform of a piecewise smooth function $f(t)$ on $\mathbb{R}_{\geq 0}$?

   A. $\frac{1}{s(s^2+1)}$

   B. $\frac{e^{-2s}}{s(s^2+4)}$

   C. $\log \frac{s^2+4}{s^2}$

   D. $\frac{2}{s^3-1}$

   E. $\frac{1}{\sqrt{s+1}}$

   F. $\frac{e^s}{s^2+1}$

   G. None of the above.

   Ans. F.

8. For $i = 1, \ldots, 6$, let $y_i(t)$ be the steady-state solution of the ordinary differential equation

$$\frac{d^2 y}{dt^2}(t) + 0.0002\frac{d\,y}{dt}(t) + 36\,y(t) = g_i(t)\,,$$

   where

$$\begin{array}{ll} g_1(t) = \cos(t) & g_2(t) = \sin^2(t) \\ g_3(t) = \cos(3t) & g_4(t) = \sin(4t) \\ g_5(t) = \cos^2(3t) & g_6(t) = \sin^2(4t) \end{array}$$

   Let $A_i = \max_{t \in \mathbb{R}} |y_i(t)|$, $i = 1, \ldots, 6$. Which one among the A_i's is the largest?

   A. A_1

   B. A_2

   C. A_3

   D. A_4

   E. A_5

   F. A_6

   G. None of the A_i's exists.

   Ans. E.

9. Let $f(x) = e^{-\frac{x^2}{2}} * x$ be the convolution of the functions $e^{-\frac{x^2}{2}}$ and x on $\mathbb{R}$. Then $f(2)$ is equal to

   A. 0

   B. $2\,e$

   C. $2\,e^{-2}$

   D. $4\,e^2$

   E. $\frac{e^{-1}}{2}$

   F. $\frac{e^{-2}}{2}$

   G. None of the above.

   Ans. C.

10. Suppose that $f(r,t)$ is a smooth function defined for all $0 \le r \le 1$, $t \ge 0$, such that

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}, \qquad f(1,t) = 0 \quad \text{for all } t \ge 0$$

and there exists two positive real numbers α, β such that $J_0(\alpha) = J_0(\beta) = 0$ and

$$f(r,0) = J_0(\alpha r), \quad \frac{\partial f}{\partial t}(r,0) = J_0(\beta r) \qquad \text{for all } r \ge 0\,.$$

Which of the following statements is true?

A. $f(0,t)$ is a periodic function in t with period $\alpha + \beta$.

B. $\lim_{t\to\infty} f(0,t) = 0$.

C. $\frac{\partial f}{\partial r}(0,0) = 0$.

D. $\int_0^1 f(r,t)\,dr$ is independent of t.

E. $f(0,t)$ is a periodic function in t with period 2π.

F. $\int_0^1 f(r,t)^2\,dr$ is independent of t.

G. None of the above.

And. C. $f(r,t) = \cos(\alpha t)\,J_0(\alpha t) + \frac{1}{\beta}\sin(\beta t)\,J_0(\beta t)$ (Actually, one does not need to solve the PDE to see that C is correct.)

11. Suppose that $u(x,t)$ is a smooth function defined for all $0 \leq x \leq \pi, t \geq 0$, which satisfies the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

and such that

$$u(0,t) = \tfrac{\partial u}{\partial x}(\pi,t) = 0 \quad \text{for all } t \geq 0$$
$$u(x,0) = 2\,\sin(\tfrac{x}{2}) - \sin(\tfrac{3x}{2}) \ \text{ for all } t \geq 0\,.$$

Then $u(\frac{\pi}{3}, 2)$ is equal to

A. $e - e^{-1}$

B. $\frac{e^{-1}}{\sqrt{3}} + \frac{e^{-4}}{4}$

C. $\frac{e}{3} - e^{-1}$

D. $\frac{\sqrt{3e}}{2} + 3e^2$

E. $e^{-1} - e^{-9}$

F. $e^{-\frac{1}{4}} + \frac{e^{-\frac{1}{3}}}{\sqrt{3}}$

G. None of the above.

Ans. E. $u(x,t) = 2\sin(\frac{x}{2})\, e^{-\frac{4}{4}} - \sin()\, e^{-\frac{9t}{4}}$

12. Let C be the circle $\{|z| = 2\}$ on the complex plane, oriented counterclockwise. Which ones of the following three integrals are equal to 0?

(1) $\displaystyle\oint_C \frac{e^z - e}{z - 1} dz$

(2) $\displaystyle\oint_C \frac{e^{z^2} - e}{z^2 - 1} dz$

(3) $\displaystyle\oint_C \frac{e^{z^3} - e}{z^3 - 1} dz$

A. Only (1)

B. Only (2)

C. Only (3)

D. Only (1) and (2)

E. Only (2) and (3)

F. Only (1) and (3)

G. (1), (2) and (3).

Ans. G.

13. Find the residue of the functions

$$f(z) = \frac{1}{e^z - 1}, \qquad g(z) = \frac{1}{e^z - 1 - z}$$

at $z = 0$.

Ans. $Res_{z=0}f(z) = 1$, $Res_{z=0}g(z) = 0$.

14. Compute the improper integral

$$\int_0^\infty \frac{\cos(2\pi x)}{1 + x^2}\, dx$$

15. Let $f(x)$ be the *even* periodic function on $\mathbb{R}$ with period 2π, such that

$$f(x) = \begin{cases} 2 & \text{for} \quad 0 \le x \le \frac{\pi}{4} \\ 0 & \text{for} \quad \frac{\pi}{4} < x \le \pi \end{cases}$$

(a) Compute the Fourier series

$$\sum_{n=0}^{\infty} a_n \cos(nx)$$

for $f(x)$.

Ans. $\frac{1}{2} - \frac{2\sqrt{2}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \cos((4k+1)x)}{4k+1} \frac{2\sqrt{2}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \cos((4k+3)x)}{4k+3} \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \cos((4k+2)x)}{4k+2}$

(b) Find $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|^2$.

Ans. $\frac{7}{4}$

16. Suppose that $f(x)$ is an *odd* periodic real-valued function on $\mathbb{R}$ with period 2π, such that the area of the region

$$\{(x, y) \in \mathbb{R}^2 \,|\, 0 \le y \le f(x)^2,\ 0 \le x \le \pi\}$$

is equal to 5π. Let $\sum_{n=1}^{\infty} b_k \sin(kx)$ be the Fourier series for $f(x)$. Find $\sum_{n=1}^{\infty} b_n^2$.

17. Find the Fourier transform of the function $f(x) = e^{-x^2}$. (Hint: Apply Cauchy's theorem to the rectangle with vertices $R, R + ia, -R + ia, -R$.)

18. Consider a thin circular metal disk 2 meters in diameter, whose faces are perfectly insulated. Suppose that the temperature at the boundary of the disk is kept at $10\cos(\theta)$, where θ is the angle formed with the horizontal line through the center of the disk. Find the steady-state temperature distribution on the disk.

19. Let α_1 be the smallest positive zero of the Bessel function $J_0(x)$. Find a solution $u(r,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$

defined for $0 \leq r \leq 1, t \geq 0$ such that $u(1,t) = 0$ for all $t \geq 0$, and $u(r,0) = 2\,J_0(\alpha_1\, r)$, and $\frac{\partial u}{\partial t}(r,0) = 5\,J_0(\alpha_1\, r)$ for all $0 \leq r \leq 1$.

20. Suppose that $u(x,t)$ is a function defined on for $0 \leq x \leq \pi, t \geq 0$, which satisfies the differential equation

$$\frac{\partial u}{\partial t} = 25\frac{\partial^2 u}{\partial x^2}$$

and such that $u(0,t) = u(\pi,t) = 0$ for all $t \geq 0$, and

$$u(x,0) = \cos(x)\,\sin(x) + \sin^3(x) \quad \text{for all } t \geq 0$$

Find the function $u(x,t)$.