NOTES ON VECTOR SPACES

The concept of *vector spaces* and the related concepts such as *linear span*, *linear independence basis* and *dimension* are of fundamental importance in mathematics. At several places in Math 240 these concepts are used, for instance on p.368 of 8.3. Here is a summary of the definitions.

An abstract vector space is a set (whose elements are called "vectors"), plus two operations, called *vector addition and scalar multiplication*. The set of scalars used in Math 240 is either the set $\mathbb{R}$ of all real numbers, or the set $\mathbb{C}$ of all complex numbers. You may be more familiar with real numbers, but in many situations you will need complex numbers. Section 7.6 of Zill and Cullen, gives a description of vector spaces over real numbers. In this note we write F for the field of scalars of a vector space over F, where F is either $\mathbb{R}$ or $\mathbb{C}$.

Definition. A vector space V over F consists of the following data: a set V, a special element $\vec{0} \in V$, and two binary operations, vector addition (denoted by $+$) and scalar multiplication (denoted by $\cdot$)

$$+ : V \times V \longrightarrow V \quad \text{and} \quad \cdot : F \times V \longrightarrow V$$

satisfying the following properties.

(i) $v + \vec{0} = \vec{0} + v = v$ for all $v \in V$.

(ii) $u + v = v + u$ for all $u, v \in V$.

(iii) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

(iv) $0 \cdot v = \vec{0}$ for all $v \in V$.

(v) $1 \cdot v = v$ for all $v \in V$.

(vi) $v + ((-1) \cdot v) = \vec{0}$ for all $v \in V$.

(vii) $\lambda \cdot (u + v) = (\lambda \cdot u) + (\lambda \cdot v)$ for all $\lambda \in F$ and all $u, v \in V$.

(viii) $(\lambda + \mu) \cdot v = (\lambda \cdot v) + (\mu \cdot v)$ for all $\lambda, \mu \in F$ and all $v \in V$.

Definition. Let V be a vector space over F.

(1) A *linear combination* of elements $u_1, \ldots, u_n \in V$ is an element of V of the form

$$\lambda_1 \cdot u_1 + \lambda_2 \cdot u_2 + \cdots + \lambda_n \cdot u_n$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$ are scalars.

(2) A subset $S \subset V$ is said to be *linearly independent* if there is no non-trivial linear relation between elements of S. In other words, if $u_1, \ldots, u_n$ are distinct elements of S, $a_1, \ldots, a_n \in F$ are n scalars and if $a_1 \cdot u_1 + a_2 \cdot u_2 + \cdots + a_n \cdot u_n = \vec{0}$, then $a_1 = a_2 = \cdots = a_n = 0$.

(3) A subset $T \subset V$ *spans* V if every element of V is a linear combination of elements of T.

The following is the one of the most important theorems in linear algebra. The concepts of *basis* and *dimension* are based on it.

Theorem. Let V be a vector space over F.

(1) Let S be a subset of V. The following are equivalent.

(a) S is a maximal linearly independent subset of V, i.e. every subset $T \supsetneqq S$ of V is linearly dependent.

(b) S is a minimal subset of V which spans V, i.e. no subset $T \subsetneqq S$ spans V.

(c) Every element of V can be written in one and only one way as a linear combination of elements of S.

A subset S of V satisfying the above equivalent conditions is called a *basis* of V.

(2) Any two bases of V have the same *cardinality*, i.e. they have the same number of elements. This number is called the *dimension* of V.

Examples.

1. The set of all column vectors with n entries in $\mathbb{R}$ form an (n-dimensional) vector space over $\mathbb{R}$. Similarly $\mathbb{C}^n$ is an (n-dimensional) vector space over $\mathbb{C}$.

2. The set of all smooth (i.e. infinitely differentiable) functions on $\mathbb{R}$ with values in $\mathbb{R}$ (respectively in $\mathbb{C}$) has a natural structure as a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$). They are usually written as $C^\infty((-\infty,\infty);\mathbb{R})$ and $C^\infty((-\infty,\infty);\mathbb{C})$ respectively. These function spaces are infinite dimensional.

3. The set of all $m \times n$ matrices with entries in $\mathbb{R}$ has a natural structure as an mn-dimensional vector space over $\mathbb{R}$. The set of all $m \times n$ matrices with entries in $\mathbb{C}$ has a natural structure as an mn-dimensional vector space over $\mathbb{C}$, which can be regarded also as a $2mn$-dimensional vector space over $\mathbb{R}$. (In particular $\mathbb{C}$ has a natural structure as a 2-dimensional vector space over $\mathbb{R}$.)

4. Let $A \in \mathrm{M}_{m,n}(F)$ be an $m \times n$ matrix with entries in F, $F = \mathbb{R}$ or $\mathbb{C}$.

   (a) The set of all F-linear combinations of the *columns* of A form a vector subspace $\mathrm{Im}(A)$ of F^m. i.e. $\mathrm{Im}(A)$ is a subset of F^n which is stable under vector addition and scalar multiplication, hence itself has a natural structure as a vector space over F. The dimension of this subspace is called the *column rank* of A.

   Similarly the dimension of of all F-linear combinations of the rows of A is a vector subspace of the space of all row vectors with n entries in F. Its dimension is called the *row rank* of A.

   Fact. The row rank and the column rank of A are equal; we call it the *rank* of A.
   (Zill and Cullen 8.3 definition 8.8 used the row rank.)

   (b) The subset $\mathrm{Ker}(A)$ of F^n consisting of all column vectors $v \in F^n$ such that $A \cdot v = \vec{0}$ is a vector subspace of F^n.

   Fact. $\dim(\mathrm{Ker}(A)) + \mathrm{rank}(A) = n$.